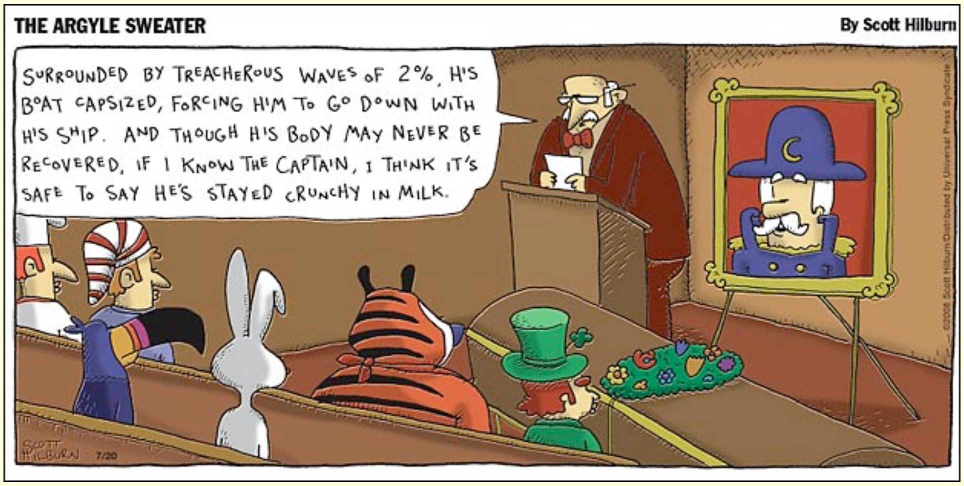
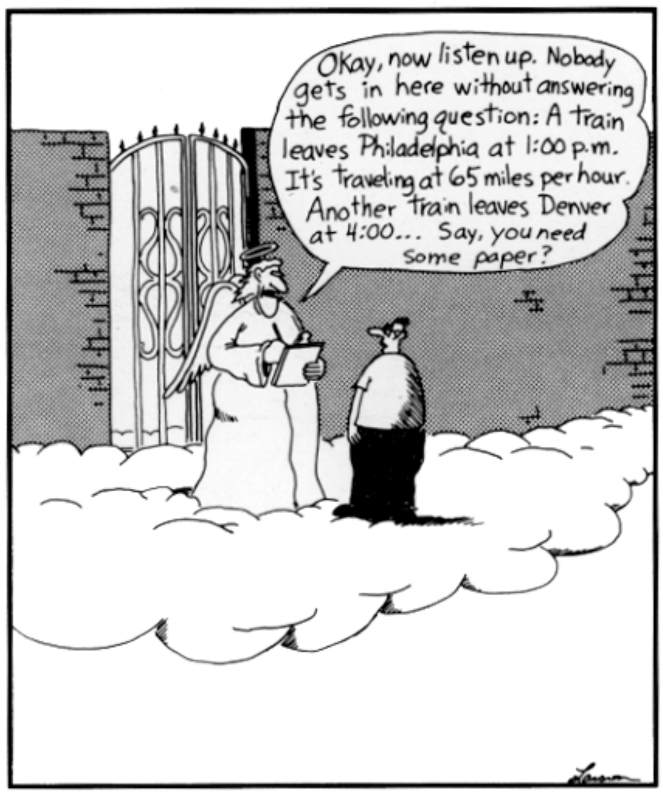
Math 5010–001: (Summer 2010)
Solutions to Sample Midterm One
(and funnies to help keep you going)



Please read the rest only after you have seriously attempted all of the problems.

1. A fair die [should be coin, sic] is cast 120 times. If all possible outcomes are equally likely, then what is the probability that there are no heads tossed?

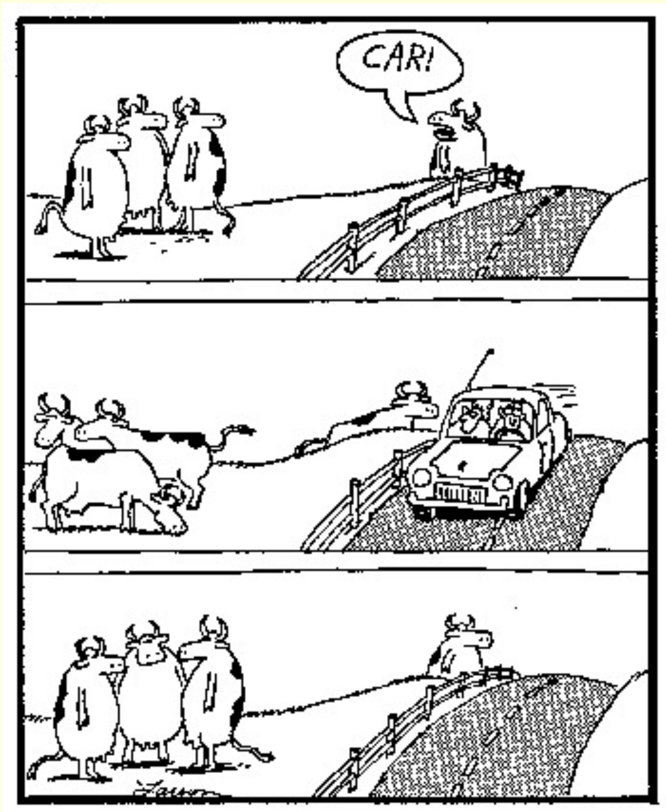


Math phobic's nightmare

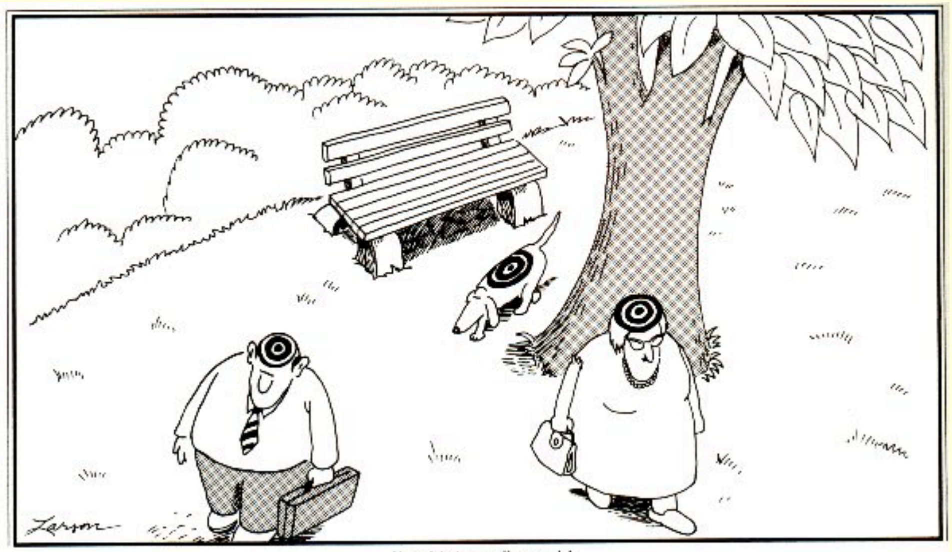
Solution: There are 2^{120} many possible outcomes. Only one of them is comprised of all tails. Therefore, $P(\text{all tails}) = 1/2^{120} = 2^{-120}$ [a *very* small, but still positive, number].

2. There are 50 men and 50 women in a room. You select 4 at random and independently [i.e., sampling with replacement]. What is the probability that the number of men in the sample is the same as the number of the women in the sample?

Solution: Let M_j denote the event that the jth drawn person is a man, and $W_j := M_j^c$ the corresponding event for a woman. Then, $P(M_1M_2W_3W_4) = P(M_1)P(M_2)P(W_3)P(W_4) = 2^{-4} = 1/16$. Similarly, $P(M_1W_2M_3W_4) = 1/16$, etc. Therefore, $P(2 \text{ men and } 2 \text{ women})$ is $1/16$ times the number of ways we can write exactly 2 M_j's in 4 spots [and therefore also 2 W_j's]. This can be computed by simply writing all the possibilities out. Or we can use the more recent lectures. Either way, the number of different ways to write 2 M's exactly in 4 spots is $\binom{4}{2} = 6$. Therefore, the chances of having 2 men and 2 women is $6 \times \frac{1}{16} = \frac{3}{8}$.



3. Four digits are selected independently at random (without repetition) from $\{0, \ldots, 9\}$. What is the probability the the four digits form a run? [For example, 0, 1, 2, 3.]



How birds see the world

Solution: We obtain a four-digit run if and only if one [and only one] of the following happens:

- We roll $0, 1, 2, 3$;
- We roll $1, 2, 3, 4$;
- We roll $2, 3, 4, 5$;

$\vdots$

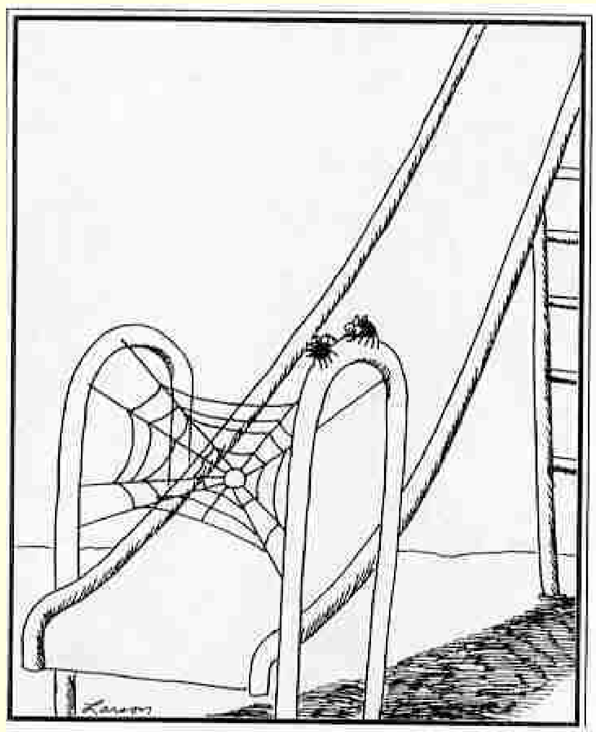
- We roll $6, 7, 8, 9$.

Each possibility has chance $(1/10)^4 = 0.001$; there are 7 possibilities [start the run with $0 \ldots 6$]. Therefore, the probability that we obtain a 4-digit run is $0.001 \times 7 = 0.007$.

4. How many different messages can be sent by using five dashes and three dots?

Solution: This problem is not going to appear in the first midterm this summer [we have no solutions to this sort of assignment]. But here is a solution any way [you will need to be familiar with this sort of problem the next midterm]: We have altogether 8 spots; we are supposed to find the number of ways of selecting the places for the dashes [say; all the remaining places are dots]. Therefore, the number of possible arrangements is

$$\binom{8}{5} = \frac{8!}{3! \cdot 5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 54.$$



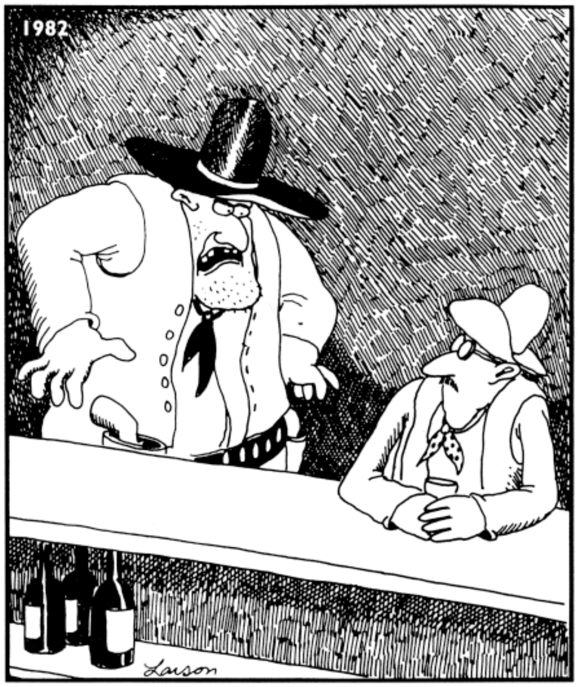
"If we pull this off, we'll eat like kings."

5. I have two coins: One is two-headed; the other is fair. I select one coin at random (both coins equally likely), and toss it N times independently. Suppose I tell you that all tosses resulted in heads. Given this information, what would you say the odds are that I had selected the fair coin?

Solution: Let F denote the event that the fair coin is selected. And H_j the event that the jth toss results in heads. We know the following: $P(F) = \frac{1}{2}$; $P(H_1 \cap \cdots \cap H_N \mid F) = (1/2)^N = 2^{-N}$; and $P(H_1 \cap \cdots \cap H_N \mid F^c) = 1$. We are asked to compute $P(F \mid H_1 \cap \cdots \cap H_N)$. We apply Bayes' rule to find that

$$\begin{aligned} & P(F \mid H_1 \cap \cdots \cap H_N) \\ & = \frac{P(H_1 \cap \cdots \cap H_N \mid F)P(F)}{P(H_1 \cap \cdots \cap H_N \mid F)P(F) + P(H_1 \cap \cdots \cap H_N \mid F^c)P(F^c)} \\ & = \frac{2^{-N} \times \frac{1}{2}}{\left(2^{-N} \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right)} = \frac{2^{-N}}{2^{-N} + 1}. \end{aligned}$$

This is the answer. But let me end with a few remarks.



"I asked you a question, buddy. ... What's the square root of 5,248?"

Note that $2^{-N}+1 \approx 1$ when N is even modestly large. For example, in the case that $N = 20$ we have $2^{-20}+1 \approx 1.0000009537$. Therefore, we can conclude among other things that

$$P(F \mid H_1 \cap \cdots \cap H_N) \approx 2^{-N} \qquad \text{for } N \text{ modestly large, or greater.}$$

In other words, if we observe N independent tosses of a coin and they were all heads, then it is very unlikely that the coin were fair.

“Mr. Osborne, may I be excused? My brain is full.”

Good luck on Monday.