Unordered Selection, continued

Let us recall the following:

Theorem 1. *The number of ways to create a team of r things among n is "n choose r." Its numerical value is*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example 1. If there are n people in a room, then they can shake hands in $\binom{n}{2}$ many different ways. Indeed, the number of possible hand shakes is the same as the number of ways we can list all pairs of people, which is clearly $\binom{n}{2}$. Here is another, equivalent, interpretation. If there are n vertices in a "graph," then there are $\binom{n}{2}$ many different possible "edges" that can be formed between distinct vertices. The reasoning is the same.

Example 2 (Recap)**.** There are $\binom{52}{5}$ many distinct poker hands.

Example 3 (Flushes in poker)**.** The number of all possible diamond flushes [all suits are diamonds] in a 5-card hand is $\binom{13}{5}$. This is also the number of heart, clubs, and spade flushes. Therefore,

$$\begin{aligned} P(\text{flush}) &= 4P(\text{diamond flush}) \\ &= 4 \times \frac{\binom{13}{5}}{\binom{52}{5}}. \end{aligned}$$

It is important that you check that this answer agrees with our previous solution to this very same problem (Example 4 on page 10).

Example 4 (Pairs in poker)**.** The number of different "pairs" $[a, a, b, c, d]$ is

$$\underbrace{13}_{\text{choose the } a} \times \underbrace{\binom{4}{2}}_{\text{deal the two } a\text{'s}} \times \underbrace{\binom{12}{3}}_{\text{choose the } b,\, c, \text{ and } d} \times \underbrace{4^3}_{\text{deal } b, c, d}.$$

Therefore,

$$P(\text{pairs}) = \frac{13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3}{\binom{52}{5}} \approx 0.42.$$

Example 5 (Poker)**.** Let A denote the event that we get two pairs $[a, a, b, b, c]$. Then,

$$|A| = \underbrace{\binom{13}{2}}_{\text{choose } a, b} \times \underbrace{\binom{4}{2}^2}_{\text{deal the } a, b} \times \underbrace{13}_{\text{choose } c} \times \underbrace{4}_{\text{deal } c}.$$

Therefore,

$$P(\text{two pairs}) = \frac{\binom{13}{2} \times \binom{4}{2}^2 \times 13 \times 4}{\binom{52}{5}} \approx 0.06.$$

Example 6. How many subsets does $\{1, \ldots, n\}$ have? Assign to each element of $\{1, \ldots, n\}$ a zero ["not in the subset"] or a one ["in the subset"]. Thus, the number of subsets of a set with n distinct elements is 2^n.

Example 7. Choose and fix an integer $r \in \{0, \ldots, n\}$. The number of subsets of $\{1, \ldots, n\}$ that have size r is $\binom{n}{r}$. This, and the preceding proves the following amusing combinatorial identity:

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$

You may need to also recall the first principle of counting.

The preceding example has a powerful generalization.

Theorem 2 (The binomial theorem)**.** *For all integers $n \geq 0$ and all real numbers x and y,*

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

Remark 1. When $n = 2$, this yields the familiar algebraic identity
$$(x + y)^2 = x^2 + 2xy + y^2.$$
For $n = 3$ we obtain
$$\begin{aligned}(x + y)^3 &= \binom{3}{0}x^0y^3 + \binom{3}{1}x^1y^2 + \binom{3}{2}x^2y^1 + \binom{3}{3}x^3y^0 \\ &= y^3 + 3xy^2 + 3x^2y + x^3.\end{aligned}$$

Proof. This is obviously correct for $n = 0, 1, 2$. We use induction. Induction hypothesis: True for $n - 1$.
$$\begin{aligned}(x + y)^n &= (x + y) \cdot (x + y)^{n-1} \\ &= (x + y)\sum_{j=0}^{n-1}\binom{n-1}{j}x^jy^{n-j-1} \\ &= \sum_{j=0}^{n-1}\binom{n-1}{j}x^{j+1}y^{n-(j+1)} + \sum_{j=0}^{n-1}\binom{n-1}{j}x^jy^{n-j}.\end{aligned}$$
Change variables [$k = j + 1$ for the first sum, and $k = j$ for the second] to deduce that
$$\begin{aligned}(x + y)^n &= \sum_{k=1}^{n}\binom{n-1}{k-1}x^ky^{n-k} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^ky^{n-k} \\ &= \sum_{k=1}^{n-1}\left\{\binom{n-1}{k-1} + \binom{n-1}{k}\right\}x^ky^{n-k} + x^n + y^n.\end{aligned}$$
But
$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left\{\frac{1}{n-k} + \frac{1}{k}\right\} \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!} \times \frac{n}{(n-k)k} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}.\end{aligned}$$
The binomial theorem follows. □