Lecture 5

Combinatorics

Recall the two basic principles of counting. These are facts that we all learn as children, as they motivate addition [in the first case] and multiplication [in the second case]:

First principle: m distinct garden forks plus n distinct fish forks equals $m+n$ distinct forks.

Second principle: m distinct knives and n distinct forks equals mn distinct ways of taking a knife and a fork.

Unordered Selection

Example 1. 6 dice are rolled at random [all possible outcomes equally likely]. What is the probability that they all show different faces?

$\Omega = ?$

$|\Omega| = 6^6$.

If A is the event in question, then $|A| = 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

Definition 1. If k is an integer ≥ 1, then we define "k factorial" as the following integer:

$$k! = k \cdot (k-1) \cdot (k-2) \cdots 2 \cdot 1.$$

For consistency of future formulas, we define also

$$0! = 1.$$

Example 2. Five rolls of a fair die. What is $P(A)$, where A is the event that all five show different faces? Note that $|A|$ is equal to 6 [which face is left out] times 6^5. Thus,

$$P(A) = \frac{6 \cdot 5!}{6^5} = \frac{6!}{6^5}.$$

Alternatively, we can use conditional probabilities to find this probability (check!) in this equivalent form: $\frac{6}{6} \times \frac{5}{6} \times \frac{4}{6} \times \frac{3}{6} \times \frac{2}{6} \times \frac{1}{6}$.

Ordered Selection

Example 3. Two-card poker.

$$P(\text{doubles}) = \frac{13 \times \left(\frac{4\times 3}{2}\right)}{\left(\frac{52\times 51}{2}\right)}.$$

Theorem 1. *n objects are divided into r types. n_1 are of type 1; n_2 of type 2; ...; n_r are of type r. Thus, $n = n_1 + \cdots + n_r$. Objects of the same type are indistinguishable. The number of permutations is*

$$\binom{n}{n_1, \ldots, n_r} = \frac{n!}{n_1! \cdots n_r!}.$$

Proof. Let N denote the number of permutations; we seek to find N. For every permtation in N there are $n_1! \cdots n_r!$ permutations wherein all n objects are treated differently. Therefore, $n_1! \cdots n_r! N = n!$. Solve to finish. □

Example 4. n people; choose r of them to form a "team." The number of different teams is then

$$\frac{n!}{r!(n-r)!}.$$

You have to choose r of type 1 ("put this one in the team"), and $n - r$ of type 2 ("leave this one out of the team").

Definition 2. We write the preceding count statistic as "n choose r," and write it as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}.$$

Example 5. Roll 4 dice; let A denote the event that all faces are different. Then,

$$|A| = \binom{6}{4} 4! = \frac{6!}{2!} = \frac{6!}{2}.$$

The 6-choose-4 is there because that is how many ways we can choose the different faces. Thus,

$$P(A) = \frac{6!}{2 \times 4^6}.$$

Example 6. There are

$$\binom{52}{5} = 2,598,960$$

different standard poker hands possible.