Lecture 4

Independence

- Events A and B are said to be *independent* if
$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

Divide both sides by $\mathrm{P}(B)$, if it is positive, to find that A and B are independent if and only if
$$\mathrm{P}(A \mid B) = \mathrm{P}(A).$$
"Knowledge of B tells us nothing new about A."

Two experiments are *independent* if A_1 and A_2 are independent for all outcomes A_j of experiment j.

Example 1. Toss two fair coins; all possible outcomes are equally likely. Let H_j denote the event that the jth coin landed on heads, and $T_j = H_j^c$. Then,
$$\mathrm{P}(H_1 \cap H_2) = \frac{1}{4} = \mathrm{P}(H_1)\mathrm{P}(H_2).$$
In fact, the two coins are independent because $\mathrm{P}(T_1 \cap T_2) = \mathrm{P}(T_1 \cap H_2) = \mathrm{P}(H_1 \cap H_2) = 1/4$ also. Conversely, if two fair coins are tossed independently, then all possible outcomes are equally likely to occur. What if the coins are not fair, say $\mathrm{P}(H_1) = \mathrm{P}(H_2) = 1/4$?

- Three events A_1, A_2, A_3 are *independent* if any two of them are, and $\mathrm{P}(A_1 \cap A_2 \cap A_3) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)$. Events A_1, A_2, A_3, A_4 are independent if any three of are, and $\mathrm{P}(A_1 \cap A_2 \cap A_3 \cap A_4) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)\mathrm{P}(A_4)$. And in general, once we have defined the independence of $n-1$ events, we define n events $A_1, \ldots, A_n$ to be *independent* if any $n-1$ of them are independent, and $\mathrm{P}(\cap_{j=1}^n A_j) = \prod_{j=1}^n \mathrm{P}(A_j)$.

- $A_1, A_2, \cdots$ are *independent* if all finite subcollection of the A_j's are independent. This condition turns out to be equivalent to the following: $A_1, \ldots, A_n$ are independent for all $n \geq 2$.
- Experiments $E_1, E_2, \ldots$ are independent if for all events $A_1, A_2, \ldots$—where A_j depends only on the outcome of E_j—$A_1, A_2, \ldots$ are independent.

Example 2 (Coin tossing)**.** Suppose 5 fair coins are tossed independently [or what is mathematically equivalent, one coin is tossed 5 times independently]. Then, the probability of tossing 5 heads is $(1/2)^5$, and this is also the probability of tossing 5 tails, the probability of $HHHHT$, etc.

Example 3 (The geometric distribution)**.** We toss a coin independently, until the first H appears. What is the probability that we need N tosses until we stop? Let H_j and T_j respectively denote the events that the jth toss yields heads [in the first case] and tails [in the second case]. Then, the probability that we seek is

$$\mathrm{P}(T_1 \cap \cdots \cap T_{N-1} \cap H_N) = \mathrm{P}(T_1)\mathrm{P}(T_2)\cdots\mathrm{P}(T_{N-1})\mathrm{P}(H_N) = \left(\frac{1}{2}\right)^N.$$

This probability vanishes geometrically fast as $N \to \infty$. More generally still, if the coin were bent so that P(heads per toss) = p, then

$$\mathrm{P}(T_1 \cap \cdots \cap T_{N_1} \cap H_N) = (1-p)^{N-1}p.$$

Example 4 (The gambler's rule)**.** A game is played successively independently until the chances are better than 50% that we have won the game at least once. If the chances of winning are p per play, then

$$\begin{aligned}\mathrm{P}\,(\text{win at least once in } n \text{ plays}) &= 1 - \mathrm{P}\,(\text{lose } n \text{ times in a row})\\ &= 1-(1-p)^n.\end{aligned}$$

Thus, we want to choose n so that $1-(1-p)^n \geq 1/2$. Equivalently, $(1-p)^n \leq 1/2$ which is itself equivalent to $n\ln(1-p) \leq -\ln 2$. In other words, we have to play at least $n(p)$ times, where

$$n(p) := \frac{\ln 2}{\ln\left(\frac{1}{1-p}\right)} \approx \frac{0.693147180559945}{\ln\left(\frac{1}{1-p}\right)}.$$

If p—the odds of winning per play—is very small, then the preceding has an interesting interpretation. Taylor's theorem tells us that for $p \approx 0$,

$$\ln\left(\frac{1}{1-p}\right) \approx p.$$

(Check!) Therefore, $n(p) \approx 0.69315/p$.

Gambler's ruin formula

You, the "Gambler," are playing independent repetitions of a fair game against the "House." When you win, you gain a dollar; when you lose, you lose a dollar. You start with k dollars, and the House starts with K dollars. What is the probability that the House is ruined before you?

Define P_j to be the conditional probability that when the game ends you have $K + j$ dollars, given that you start with j dollars initially. We want to find P_k.

Two easy cases are: $P_0 = 0$ and $P_{k+K} = 1$.

By Theorem 1 and independence,

$$P_j = \frac{1}{2}P_{j+1} + \frac{1}{2}P_{j-1} \qquad \text{for } 0 < j < k + K.$$

In order to solve this, write $P_j = \frac{1}{2}P_j + \frac{1}{2}P_j$, so that

$$\frac{1}{2}P_j + \frac{1}{2}P_j = \frac{1}{2}P_{j+1} + \frac{1}{2}P_{j-1} \qquad \text{for } 0 < j < k + K.$$

Multiply both side by two and solve:

$$P_{j+1} - P_j = P_j - P_{j-1} \qquad \text{for } 0 < j < k + K.$$

In other words,

$$P_{j+1} - P_j = P_1 \qquad \text{for } 0 < j < k + K.$$

This is the simplest of all possible "difference equations." In order to solve it you note that, since $P_0 = 0$,

$$\begin{aligned} P_{j+1} &= (P_{j+1} - P_j) + (P_j - P_{j-1}) + \cdots + (P_1 - P_0) \qquad \text{for } 0 < j < k + K \\ &= (j+1)P_1 \qquad \text{for } 0 < j < k + K. \end{aligned}$$

Apply this with $j = k + K - 1$ to find that

$$1 = P_{k+K} = (k + K)P_1, \quad \text{and hence} \quad P_1 = \frac{1}{k + K}.$$

Therefore,

$$P_{j+1} = \frac{j + 1}{k + K} \qquad \text{for } 0 < j < k + K.$$

Set $j = k - 1$ to find the following:

Theorem 1 (Gambler's ruin formula)**.** *If you start with k dollars, then the probability that you end with $k + K$ dollars before losing all of your initial fortune is $k/(k + K)$ for all $1 \le k \le K$.*

Conditional probabilities as probabilities

Suppose B is an event of positive probability. Consider the conditional probability distribution, $\mathrm{Q}(\cdots) = \mathrm{P}(\cdots \mid B)$.

Theorem 2. Q *is a probability on the new sample space* B*. [It is also a probability on the larger sample space* Ω*, why?]*

Proof. Rule 1 is easy to verify: For all events A,

$$0 \leq \mathrm{Q}(A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} \leq \frac{P(B)}{\mathrm{P}(B)} = 1,$$

because $A \cap B \subseteq B$ and hence $\mathrm{P}(A \cap B) \leq \mathrm{P}(B)$.

For Rule 2 we check that

$$\mathrm{Q}(B) = \mathrm{P}(B \mid B) = \frac{\mathrm{P}(B \cap B)}{\mathrm{P}(B)} = 1.$$

Next suppose $A_1, A_2, \ldots$ are disjoint events. Then,

$$\mathrm{Q}\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{1}{\mathrm{P}(B)} \mathrm{P}\left(\bigcup_{n=1}^{\infty} A_n \cap B\right).$$

Note that $\cup_{n=1}^{\infty} A_n \cap B = \cup_{n=1}^{\infty} (A_n \cap B)$, and $(A_1 \cap B), (A_2 \cap B), \ldots$ are disjoint events. Therefore,

$$\mathrm{Q}\left(\bigcup_{n=1}^{\infty} A_n\right) = \frac{1}{\mathrm{P}(B)} \sum_{N=1}^{\infty} \mathrm{P}\left(A_n \cap B\right) = \sum_{n=1}^{\infty} \mathrm{Q}(A_n).$$

This verifies Rule 4, and hence Rule 3. □