Continuous joint distributions (continued)

Example 1 (Uniform distribution on the triangle)**.** Consider the random vector $(X\,,Y)$ whose joint distribution is

$$f(x\,,y)=\begin{cases}2 & \text{if } 0\le x<y\le 1,\\ 0 & \text{otherwise.}\end{cases}$$

This is a density function [on a triangle].

(1) What is the distribution of X? How about Y?

We have

$$f_X(a)=\int_{-\infty}^{\infty} f(a\,,y)\,dy.$$

If $a\notin(0\,,1)$, then $f(a\,,y)=0$ regardless of the value of y [draw a picture!]. Therefore, for $a\notin(0\,,1)$, $f_X(a)=0$. If on the other hand $0<a<1$, then [draw a picture!],

$$f_X(a)=\int_a^1 2\,dy=2(1-a).$$

That is,

$$f_X(a)=\begin{cases}2(1-a) & \text{if } 0<a<1,\\ 0 & \text{otherwise.}\end{cases}$$

Similarly,

$$f_Y(b)=\int_0^b 2\,dx=2b \qquad \text{if } 0<b<1,$$

and $f_Y(b)=0$, otherwise.

(2) Are X and Y independent?

No, there exist [many] choices of $(x\,,y)$ such that $f(x\,,y) = 2 \neq f_X(x)f_Y(y)$. In fact, $P\{X < Y\} = \iint f = 1$ [check!].

(3) Find EX and EY. Also compute the SDs of X and Y.

Let us start with the means:

$$EX = \int_0^1 x \overbrace{2(1-x)}^{f_X(x)}\, dx = 2\int_0^1 x\, dx - 2\int_0^1 x^2\, dx = \frac{1}{3};$$

similarly,

$$EY = \int_0^1 y \overbrace{2y}^{f_Y(y)}\, dy = \frac{2}{3}.$$

Also:

$$E(X^2) = \int_0^1 x^2 2(1-x)\, dx = \frac{1}{6} \quad\Rightarrow\quad \mathrm{Var}X = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

Similarly,

$$E(Y^2) = \int_0^1 y^2 2y\, dy = \frac{1}{2} \quad\Rightarrow\quad \mathrm{Var}(Y) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}.$$

Consequently, $\mathrm{SD}(X) = \mathrm{SD}(Y) = 1/\sqrt{18}$.

(4) Compute $E(XY)$.

After we draw a picture [of the region of integration], we find that

$$E(XY) = \int_0^1 \int_x^1 2xy\, dy\, dx = 2\int_0^1 y\left(\int_0^y x\, dx\right) dy = 2\int_0^1 \frac{1}{2}y^3\, dy = \frac{1}{4}.$$

(5) Define correlation as in the discrete. Then what is the correlation between X and Y?

The correlation is

$$\rho := \frac{E(XY) - EXEY}{\mathrm{SD}(X)\mathrm{SD}(Y)} = \frac{\frac{1}{4} - \left(\frac{1}{3}\times\frac{2}{3}\right)}{\frac{1}{\sqrt{18}}\times\frac{1}{\sqrt{18}}} = \frac{1}{2}.$$

The distribution of a sum

Suppose $(X\,,Y)$ has joint density $f(x\,,y)$. Question: What is the distribution of $X+Y$ in terms of the function f?

$$F_{X+Y}(a) = P\{X + Y \le a\} = \int_{-\infty}^{\infty} \int_{-\infty}^{-x+a} f(x\,, y)\, dy\, dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a} f(x\,, z - x)\, dz\, dx.$$

Differentiate $[d/da]$ to obtain the density of $X + Y$, using the fundamental theorem of calculus:

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f(x\,, a - x)\, dx.$$

An important special case: X and Y are *independent* if $f(x\,, y) = f_X(x)f_Y(y)$ for all pairs $(x\,, y)$. If X and Y are independent, then

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(x)f_Y(a - x)\, dx.$$

This is called the *convolution* of the functions f_X and f_Y.

Example 2. Suppose X and Y are independent exponentially-distributed random variables with common parameter λ. What is the distribution of $X + Y$?

We know that $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f_X(x) = 0$ otherwise. And f_Y is the same function as f_X. Therefore,

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(x)f_Y(a - x)\, dx$$
$$= \int_{0}^{\infty} \lambda e^{-\lambda x} f_Y(a - x)\, dx = \int_{0}^{a} \lambda e^{-\lambda x} \lambda e^{-\lambda(a-x)}\, dx$$
$$= \lambda^2 a e^{-\lambda a},$$

provided that $a > 0$. And $f_{X+Y}(a) = 0$ if $a \le 0$. In other words, the sum of two independent exponential (λ) random variables has a gamma density with parameters $(2\,,\lambda)$. We can generalize this (how?) as follows: If $X_1, \ldots, X_n$ are independent exponential random variables with common parameter $\lambda > 0$, then $X_1 + \cdots + X_n$ has a gamma distribution with parameters $r = n$ and λ. A special case, in applications, is when $\lambda = \frac{1}{2}$. A gamma distribution with parameters $r = n$ and $\lambda = \frac{1}{2}$ is also known as a χ^2 distribution [pronunced "chi squared"] with n "degrees of freedom." This distribution arises in many different settings, chief among them in multivariable statistics and the theory of continuous-time stochastic processes. □

The distribution of a sum (discrete case)

It is important to understand that the preceding "convolution formula" is a procedure that we ought to understand easily when X and Y are discrete instead.

Example 3 (Two draws at random, Pitman, p. 144)**.** We make two draws at random, without replacement, from a box that contains tickets numbered 1, 2, and 3. Let X denote the value of the first draw and Y the value of the second draw. The following tabulates the function $f(x\,,y) = P\{X = x\,,Y = y\}$ for all possible values of x and y:

		possible value for X		
		1	2	3
possible	3	1/6	1/6	0
values	2	1/6	0	1/6
for Y	1	0	1/6	1/6

We want to know the distribution of $X + Y$ = the total number of dots rolled. Here is a way to compute that: First of all, the possible values of $X + Y$ are $3, 4, 5$. Next, we note that

$$P\{X + Y = 3\} = P\{X = 2\,, Y = 1\} + P\{X = 1\,, Y = 1\} = \frac{1}{3},$$
$$P\{X + Y = 4\} = P\{X = 1\,, Y = 3\} + P\{X = 3\,, Y = 1\} = \frac{1}{3},$$
$$P\{X + Y = 5\} = P\{X = 2\,, Y = 3\} + P\{X = 3\,, Y = 2\} = \frac{1}{3}.$$

The preceding example can be generalized: If $(X\,,Y)$ are distributed as a discrete random vector, then

$$P\{X + Y = a\} = \sum_x P\{X = x\ ,\ Y = a - x\};$$

When X and Y are independent, the preceding simplifies to

$$\boxed{P\{X + Y = a\} = \sum_x P\{X = x\} \cdot P\{Y = a - x\};}$$

This is a "discrete convolution" formula.

The distribution of a ratio

The preceding ideas can be used to answer other questions as well. For instance, suppose $(X\,,Y)$ is jointly distributed with joint density $f(x\,,y)$. Then what is the density of Y/X?

We proceed as we did for sums:

$$\begin{aligned} F_{Y/X}(a) &= P\left\{\frac{Y}{X} \le a\right\} \\ &= P\left\{\frac{Y}{X} \le a\ ,\ Y > 0\right\} + P\left\{\frac{Y}{X} \le a\ ,\ Y < 0\right\} \\ &= P\{Y \le aX\ ,\ X > 0\} + P\{Y \ge aX\ ,\ X < 0\} \\ &= \int_0^\infty \int_{-\infty}^{ax} f(x\,,y)\,dy\,dx + \int_{-\infty}^0 \int_{ax}^\infty f(x\,,y)\,dy\,dx \\ &= \int_0^\infty \int_{-\infty}^{a} f(x\,,zx)\,xdz\,dx + \int_{-\infty}^0 \int_{a}^\infty f(x\,,zx)\,xdz\,dx. \end{aligned}$$

Differentiate, using the fundamental theorem of calculus, to arrive at

$$\begin{aligned} f_{Y/X}(a) &= \int_0^\infty f(x\,,ax)\,x\,dx - \int_{-\infty}^0 f(x\,,ax)x\,dx \\ &= \int_{-\infty}^\infty f(x\,,ax)|x|\,dx. \end{aligned}$$

In the important special case that X and Y are independent, this yields the following formula:

$$f_{Y/X}(a) = \int_{-\infty}^\infty f_X(x) f_Y(ax)|x|\,dx.$$

Example 4. Suppose X and Y are independent exponentially-distributed random variables with respective parameters α and β. Then what is the density of Y/X? The answer is

$$\begin{aligned} f_{Y/X}(a) &= \int_0^\infty \alpha e^{-\alpha x} f_Y(ax)x\,dx \\ &= \int_0^\infty \alpha e^{-\alpha x} \beta e^{-\beta a x} x\,dx \qquad [\text{if } a > 0\text{; else, } f_{Y/X}(a) = 0] \\ &= \alpha\beta \int_0^\infty x e^{-(\alpha+\beta a)x}\,dx \\ &= \frac{\alpha\beta}{(\alpha+\beta a)^2} \cdot \int_0^\infty y e^{-y}\,dy \qquad [y := (\alpha+\beta a)x] \\ &= \frac{\alpha\beta}{(\alpha+\beta a)^2} \cdot \Gamma(2) = \frac{\alpha\beta}{(\alpha+\beta a)^2}, \end{aligned}$$

for $a > 0$ and $f_{Y/X}(a) = 0$ for $a \leq 0$. In the important case that $\alpha = \beta$, we have

$$f_{Y/X}(a) = \begin{cases} \dfrac{1}{(1+a)^2} & \text{if } a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note, in particular, that

$$E\left(\frac{Y}{X}\right) = \int_0^\infty \frac{a}{(1+a)^2}\, da = \infty.$$

Example 5. Suppose X and Y are independent standard normal random variables. Then a similar computation shows that

$$f_{Y/X}(a) = \frac{1}{\pi(1+a^2)} \qquad \text{for all real } a.$$

[See Example 5, p. 383 of your text.] This is called the *standard Cauchy density*. Note that the Cauchy density does not have a well-defined expectation, although

$$E\left(\left|\frac{Y}{X}\right|\right) = \frac{1}{\pi} \cdot \int_{-\infty}^\infty \frac{|a|}{1+a^2}\, da = \frac{2}{\pi} \cdot \int_0^\infty \frac{a}{1+a^2}\, da = \infty.$$

Exercise. One might wish to know about the distribution of Y/X when Y and X are discrete random variables. Check that if X and Y are discrete and $P\{X = 0\} = 0$, then

$$P\left\{\frac{Y}{X} = a\right\} = \sum_{x \neq 0} P\{X = x\} \cdot P\{Y = ax\}.$$

Note that if we replace the sum by an integral and probabilities with densities we do *not* obtain the correct formula for continuous random variables [$|x|$ is missing!].

Functions of a random vector

Basic problem: If (X, Y) has joint density f, then what, if any, is the joint density of (U, V), where $U = u(X, Y)$ and $V = v(X, Y)$? Or equivalently, $(U, V) = T(X, Y)$, where

$$T(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

Example 6. Let (X, Y) be distributed uniformly in the circle of radius $R > 0$ about the origin in the plane. Thus,

$$f_{X,Y}(x\,,y) = \begin{cases} \dfrac{1}{\pi R^2} & \text{if } x^2 + y^2 \le R^2, \\ 0 & \text{otherwise.} \end{cases}$$

We wish to write (X, Y), in polar coordinates, as (R, Θ), where

$$R = \sqrt{X^2 + Y^2} \quad \text{and} \quad \Theta = \arctan(Y/X).$$

Then, we compute first the *joint distribution function* $F_{R,\Theta}$ of (R, Θ) as follows:

$$\begin{aligned} F_{R,\Theta}(a\,,b) &= \mathrm{P}\{R \le a\,, \Theta \le b\} \\ &= \mathrm{P}\{(X, Y) \in A\}, \end{aligned}$$

where A is the "partial cone" $\{(x\,,y) : x^2 + y^2 \le a^2\,, \arctan(y/x) \le b\}$. If a is not between 0 and R, or $b \notin (-\pi\,,\pi)$, then $F_{R,\Theta}(a\,,b) = 0$. Else,

$$\begin{aligned} F_{R,\Theta}(a\,,b) &= \iint_A f_{X,Y}(x\,,y)\,dx\,dy \\ &= \int_0^b \int_0^a \frac{1}{\pi R^2} r\,dr\,d\theta, \end{aligned}$$

after the change of variables $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. Therefore, for all $a \in (0\,,R)$ and $b \in (-\pi\,,\pi)$,

$$F_{R,\Theta}(a\,,b) = \begin{cases} \dfrac{a^2 b}{2\pi R^2} & \text{if } 0 < a < R \text{ and } -\pi < b < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$f_{R,\Theta}(a\,,b) = \frac{\partial^2 F_{R,\Theta}}{\partial a \partial b}(a\,,b).$$

Therefore,

$$f_{R,\Theta}(a\,,b) = \begin{cases} \dfrac{a}{\pi R^2} & \text{if } 0 < a < R \text{ and } -\pi < b < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

The previous example can be generalized.

Suppose T is invertible with inverse function

$$T^{-1}(u\,,v) = \begin{pmatrix} x(u\,,v) \\ y(u\,,v) \end{pmatrix}.$$

The *Jacobian* of this transformation is

$$J(u\,,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 1. *If T is "nice," then*

$$f_{U,V}(u\,,v) = f_{X,Y}(x(u\,,v)\,,y(u\,,v))|J(u\,,v)|.$$

Example 7. In the polar coordinates example($r = u$, $\theta = v$),

$$\begin{aligned} r(x\,,y) &= \sqrt{x^2+y^2}, \\ \theta(x\,,y) &= \arctan(y/x) = \theta, \\ x(r\,,\theta) &= r\cos\theta, \\ y(r\,,\theta) &= r\sin\theta. \end{aligned}$$

Therefore, for all $r > 0$ and $\theta \in (-\pi\,,\pi)$,

$$\begin{aligned} J(r\,,\theta) &= (\cos(\theta) \times r\cos(\theta)) - (-r\sin(\theta) \times \sin(\theta)) \\ &= r\cos^2(\theta) + u\sin^2(\theta) = r. \end{aligned}$$

Hence,

$$f_{R,\Theta}(r\,,\theta) = \begin{cases} r f_{X,Y}(r\cos\theta\,,r\sin\theta) & \text{if } r > 0 \text{ and } \pi < \theta < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

You should check that this yields Example 6, for instance.

Example 8. Let us compute the joint density of $U = X$ and $V = X + Y$. Here,

$$\begin{aligned} u(x\,,y) &= x \\ v(x\,,y) &= x + y \\ x(u\,,v) &= u \\ y(u\,,v) &= v - u. \end{aligned}$$

Therefore,

$$J(u\,,v) = (1 \times 1) - (0 \times -1) = 1.$$

Consequently,

$$f_{U,V}(u\,,v) = f_{X,Y}(u\,,v-u).$$

This has an interesting by-product: The density function of $V = X + Y$ is

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} f_{U,V}(u\,,v)\,du \\ &= \int_{-\infty}^{\infty} f_{X,Y}(u\,,v-u)\,du. \end{aligned}$$