Lecture 15

Cumulative distribution functions

Given a random variable X, the *cumulative distribution function*—also known as the cdf—F of X is the function defined by

$$F(a) = P\{X \leq a\}.$$

If X has a continuous distribution with density function f, then

$$F(a) = \int_{-\infty}^{a} f.$$

And by the fundamental theorem of calculus, we can compute f from F as well; namely,

$$f(a) = F'(a).$$

Example 1. Let $X_1, \ldots, X_n$ be independent exponentially-distributed random variables with respective parameters $\lambda_1, \ldots, \lambda_n$. What is the distribution of $Y := \min(X_1, \ldots, X_n)$?

Note that for all $y > 0$,

$$\begin{aligned} 1 - F_Y(y) &= P\{X_1 > y, \ldots, X_n > y\} = P\{X_1 > y\} \cdots P\{X_n > y\} \\ &= e^{-\lambda_1 y} \cdots e^{-\lambda_n y} = e^{-\theta y}, \end{aligned}$$

where $\theta := \lambda_1 + \cdots + \lambda_n$. And $F_Y(y) = 0$ if $y \leq 0$. Differentiate $[d/dy]$ to see that $f_Y(y) = \theta e^{-\theta y}$ if $y > 0$ and 0 if $y \leq 0$. Thus, Y is exponentially distributed with parameter $\theta := \lambda_1 + \cdots + \lambda_n$.

Change of variables

Example 2. Suppose X has the exponential distribution with $\lambda = 1$; i.e., X has density function $f_X(a) = e^{-a}$ for $a > 0$. Set $Y := \sqrt{X}$. What is the density function f_Y of Y?

Clearly, $f_Y(a) = 0$ if $a < 0$. Key observation: If F_Y is the cdf of Y, then

$$P\{Y \le a\} = \int_0^a f_Y \quad \Rightarrow \quad \frac{d}{da} P\{Y \le a\} = f_Y(a),$$

thanks to the fundamental theorem of calculus. Now, densities are not probabilities. Therefore, they do not follow the rules of probabilities. But cdf's are genuine probabilities. Now,

$$F_Y(a) = P\{Y \le a\} = P\left\{\sqrt{X} \le a\right\} = P\left\{X \le a^2\right\} = 1 - e^{-a^2}.$$

Therefore, if $a > 0$ then

$$f_Y(a) = \frac{d}{da}\left(1 - e^{-a^2}\right) = 2ae^{-a^2}.$$

Proposition 1. *Suppose X has density function f_X on the range $(a\,,b)$. Let $Y = g(X)$ where g is either strictly increasing or strictly decreasing on $(a\,,b)$. The range of Y is then the interval with endpoints $g(a)$ and $g(b)$. And the density of Y is*

$$f_Y(y) = \frac{f_X\left(g^{-1}(y)\right)}{|g'(g^{-1}(y))|} \qquad \text{for } a < y < b.$$

Proof. We follow the strategy of the preceding example. Suppose g is strictly increasing. Then,

$$F_Y(y) = P\left\{X \le g^{-1}(y)\right\} = F_X\left(g^{-1}(y)\right) \qquad \text{for } a < y < b.$$

Therefore,

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \times \frac{d}{dy}\left(g^{-1}(y)\right),$$

and the proposition follows from implicit differentiation: Set $y = g(x)$ [equivalently, $x = g^{-1}(y)$] and note that

$$1 = g'(x)\frac{dx}{dy} \qquad \Rightarrow \frac{dx}{dy} = \frac{1}{g'(x)} = \frac{1}{g'(g^{-1}(y))}.$$

Because $dx/dy = \frac{d}{dy}g^{-1}(y)$, it follows that

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \times \frac{1}{g'(g^{-1}(y))} = \frac{f_X\left(g^{-1}(y)\right)}{|g'(g^{-1}(y))|},$$

since g is increasing, whence $g'(\alpha) = |g'(\alpha)|$ for all α. The case that g is strictly decreasing is similar, except the very first line is changed as follows:

$$F_Y(y) = P\{Y \le y\} = P\left\{X \ge g^{-1}(y)\right\} = 1 - F_X\left(g^{-1}(y)\right) \qquad \text{for } a < y < b.$$

The remainder is proved in parallel with the case that g is increasing. □

One can frequently find $f_{g(X)}$ when g is many-to-one as well; see pp. 306–307 of your text.

Continuous joint distributions

Two random variables X and Y, defined both on the same probability space, are said to be jointly distributed with joint density f if

$$P\{(X\,,Y) \in A\} = \iint_A f.$$

Here, the "joint density function" f is a function of two variables $[f(x\,,y)]$.[1]

The defining properties of f are:

$$f(x\,,y) \ge 0 \qquad \text{for all } x, y, \text{ and} \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f = 1.$$

The theory of several continuous random variables is very similar to the analogous discrete theory. For instance, if $g(x\,,y)$ is a function of two variables, then

$$Eg(X\,,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x\,,y) f(x\,,y)\,dx\,dy,$$

provided that either $g(x\,,y) \ge 0$ or $\iint |g(x\,,y)| f(x\,,y)\,dx\,dy < \infty$.

As in the discrete theory, we can find the density of X and the density of Y from the joint density f. For example, because

$$F_X(a) = P\{X \le a\} = P\{X \le a\,, Y < \infty\} = \int_{-\infty}^{a}\int_{-\infty}^{\infty} f(x\,,y)\,dy\,dx,$$

it follows from the fundamental theorem of calculus that

$$f_X(a) = F_X'(a) = \frac{d}{da}\int_{-\infty}^{a}\left(\int_{-\infty}^{\infty} f(x\,,y)\,dy\right) dx = \int_{-\infty}^{\infty} f(a\,,y)\,dy.$$

And similarly,

$$f_Y(a) = \int_{-\infty}^{\infty} f(x\,,a)\,dx.$$

Finally, X and Y are independent [i.e., $P\{X \in A\,, Y \in B\} = P\{X \in A\}P\{Y \in B\}$] if and only if $f(x\,,y) = f_X(x) f_Y(y)$ for all pairs $(x\,,y)$. As was the case for

[1] You should *always* plot the region of integration for double integrals!

discrete random vectors, if X and Y are independent, then $E[g_1(X)g_2(Y)] = E[g_1(X)] \cdot E[g_2(X)]$, whenever the expectations are defined, and absolutely convergent [as integrals].

Example 3 (Uniform distribution on the square)**.** Consider the random vector $(X\,,Y)$ whose joint distribution is

$$f(x\,,y) = \begin{cases} 1 & \text{if } 0 \le x, y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) What is $P\{X \le Y\}$? Write this as $P\{(X\,,Y) \in A\}$ and plot A to find that

$$P\{X \le Y\} = \int_0^1 \int_0^y dx\,dy = \int_0^1 y\,dy = \frac{1}{2}.$$

Similarly [draw a picture!],

$$P\left\{X \le \frac{Y}{2}\right\} = \int_0^1 \int_0^{y/2} dx\,dy = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}.$$

(2) What is the distribution of X? We compute the density:

$$f_X(a) = \int_0^1 f(a\,,y)\,dy = \begin{cases} 1 & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, X is distributed uniformly on $(0\,,1)$. And so is Y [check!].

(3) Are X and Y independent? Yes; indeed, $f(x\,,y) = f_X(x)f_Y(y)$ for all pairs $(x\,,y)$.

(4) Find $E(X)$, $E(Y)$, and $E(\sqrt{X/Y})$.

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\,dx = \frac{1}{2} = E(Y).$$

And

$$E(\sqrt{X/Y}) = \int_0^1 \int_0^1 \sqrt{\frac{x}{y}}\,dx\,dy = \int_0^1 \frac{1}{\sqrt{y}} \left(\int_0^1 \sqrt{x}\,dx \right) dy.$$

The integral in the brackets is $\frac{2}{3}$. Therefore,

$$E(\sqrt{X/Y}) = \frac{2}{3} \int_0^1 \frac{1}{\sqrt{y}}\,dy = \frac{4}{3}.$$