Lecture 14

Continuous distributions

Some physical examples of random variables that arise which do not have a "discrete distribution":

- Sample N people at random; let $X :=$ the average weight [in pounds] in the sample;
- Sample a person at random; compute their blood pressure; prescribe a dose of your blood-pressue medicine; compute the blood pressure a day after the medicine was taken. Let $X :=$ the difference [after minus before] in blood pressure; etc.

Mathematics model for a random variable X that has a "continuous distribution": There exists a *probability density* [call it f here] which is a function such that

$$P\{a \le X \le b\} = \int_a^b f(x)\,dx \qquad \text{for all } a \le b.$$

In fact, the preceding is equivalent to the statement that

$$P\{X \in A\} = \int_A f(x)\,dx,$$

for all sets A for which the definite integral is defined.

Proposition 1. *A function f is a probability density of some random variables if and only if $f(x) \ge 0$ for all x, and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.*

The expectation of a continuous random variable

If X has probability density function f, then the *expectation* EX is defined as

$$EX := \int_{-\infty}^{\infty} xf(x)\,dx,$$

provided that the integral is defined. The calculus of functions of one variable guarantees that if $\int_{-\infty}^{\infty} |x|f(x)\,dx$ is finite, then $\int_{-\infty}^{\infty} xf(x)\,dx$ is well defined.

Proposition 2. *Suppose g is a function, and X is a random variable with density function f. Then,*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx,$$

provided that $\int_{-\infty}^{\infty} |g(x)|f(x)\,dx < \infty$.

Therefore,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

The integrand is nonnegative; therefore the integral is always defined, but might be infinite.

The basic properties of expectations is the same in the continuous case as in the discrete case. For instance, the *variance* $\mathrm{Var}X$ of X is defined as

$$\mathrm{Var}X := E\left[(X - EX)^2\right] = E(X^2) - [EX]^2.$$

The uniform distribution

A random variable X is said to have the *uniform distribution on* $(a\,,b)$, if its density function is

$$f(x) = \begin{cases} 1/(b-a) & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $I \subset (a\,,b)$ is any interval, then

$$P\{X \in I\} = \int_I \frac{1}{b-a}\,dx = \frac{|I|}{b-a}.$$

For instance, suppose $(a\,,b) = (0\,,1)$. Then the probability of falling in I is the same as the probability of falling in J—for every two intervals I and J in $(0\,,1)$—provided that I and J have the same length. This motivates the term "uniform distribution."

If X has the uniform distribution on $(a\,,b)$, then

$$EX = \int_a^b x\frac{1}{b-a}\,dx = \frac{1}{b-a} \times \frac{1}{2} \times \left(b^2 - a^2\right) = \frac{b+a}{2},$$

since $b^2 - a^2 = (b-a)(b+a)$. Similarly,

$$E(X^2) = \int_a^b x^2\frac{1}{b-a}\,dx = \frac{1}{b-a} \times \frac{1}{3}\left(b^3 - a^3\right) = \frac{b^2+ab+a^2}{3},$$

since $b^3 - a^3 = (b-a)(b^2+ab+a^2)$, as you check my directly multiplying out the right-hand side. Consequently,

$$\begin{aligned}\text{Var}X = E(X^2) - (EX)^2 &= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 \\ &= \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} \\ &= \frac{4(b^2+ab+a^2) - 3(a^2+2ab+b^2)}{12} \\ &= \frac{b^2-2ab+a^2}{12} = \frac{(a+b)^2}{12} \qquad \Rightarrow \qquad \text{SD}(X) = \frac{a+b}{\sqrt{12}}.\end{aligned}$$

The exponential distribution

Choose and fix a number $\lambda > 0$. A random variable X has the *exponential distribution* with parameter λ if its density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In order for this definition to be coherent we need: (i) $f(x) \geq 0$ [this is fine!]; and (ii) $\int_{-\infty}^{\infty} f = 1$. Part (ii) should be verified.

Note that

$$EX = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}\int_0^\infty ze^{-z}\,dz \qquad [z := \lambda x].$$

Integrate the latter integral by parts to find that

$$EX = \frac{1}{\lambda}\left[\left(-ze^{-z}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-z}\right)dz\right] = \frac{1}{\lambda}.$$

Let me stress that, among other things, we just also found the identity:

$$\int_0^\infty ze^{-z}\,dz = 1. \tag{11}$$

Likewise,

$$E(X^2) = \int_0^\infty x^2\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda^2}\int_0^\infty z^2e^{-z}\,dz \qquad [z := \lambda x].$$

Integrate by parts to see that

$$\int_0^\infty z^2 e^{-z}\,dz = (-z^2e^{-z})\Big|_0^\infty - \int_0^\infty -2ze^{-z}\,dz = 2\int_0^\infty ze^{-z}\,dz = 2, \quad (12)$$

thanks to (11). Therefore,

$$E(X^2) = \frac{2}{\lambda^2} \qquad \Rightarrow \qquad \mathrm{Var}X = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \qquad \Rightarrow \qquad \mathrm{SD}(X) = \frac{1}{\lambda}.$$

The exponential distribution has a peculiar, and very important, property. Note that if X has the exponential distribution with parameter λ, then for all $a > 0$,

$$P\{X > a\} = \int_a^\infty f(x)\,dx = \lambda\int_a^\infty e^{-\lambda x}\,dx = e^{-\lambda a}.$$

Consequently, for all $a, b > 0$,

$$P(X > a + b \mid X > a) = \frac{P\{X > a + b\}}{P\{X > a\}} = e^{-\lambda b} = P\{X > b\}.$$

This is called the "memoryless property" of the exponential distribution.

Aside: The gamma function

Then we might ask for the numerical value of

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

where $\alpha > 0$ is arbitrary. Because $x^{-1}e^{-x} \approx x^{-1}$ for $x \approx 0$, we can deduce that $\Gamma(0) = \infty$. And it is clear that $\Gamma(1) = 1$. Furthermore, $\Gamma(2) = 1$ and $\Gamma(3) = 2$ thanks respectively to (11) and (12).

Proposition 3 (A duplication formula)**.** *For every $\alpha > 0$,*

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

Proof. We integrate by parts:

$$\Gamma(\alpha + 1) = (-x^{\alpha+1}e^{-x})\Big|_0^\infty - \int_0^\infty (-\alpha x^\alpha e^{-x})\,dx = \alpha\int_0^\infty x^\alpha e^{-x}\,dx,$$

which is $\alpha\Gamma(\alpha)$. □

As a consequence of the duplication formula for the gamma function we find that

$$\Gamma(4) = 3\Gamma(3) = 3\times 2,\ \Gamma(5) = 4\Gamma(4) = 4\times 3\times 2\ldots,$$

and $\Gamma(k) = (k-1)!$ for every integer $k \geq 1$ [induction].

Next let us note, a la J. Stirling , that

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x}\, dx = \sqrt{2} \cdot \int_0^\infty e^{-y^2/2}\, dy \qquad \left[y := \sqrt{2x}\right].$$

And the latter integral is, by symmetry, $\frac{1}{2}\int_{-\infty}^{\infty} e^{-y^2/2}\, dy = \frac{1}{2}\sqrt{2\pi}$. Therefore,

$$\Gamma(1/2) = \sqrt{\pi} \quad \Rightarrow \quad \Gamma(3/2) = \frac{\sqrt{\pi}}{2},\ \Gamma(5/2) = \frac{3\sqrt{\pi}}{4}, \ldots .$$

The remaining values of $\Gamma(\alpha)$ cannot be evaluated exactly.

The gamma distribution

Suppose $\lambda, r > 0$ are fixed. Then we say that X has the *gamma distribution* with parameters λ and r if the density function of X is

$$f(x) = \begin{cases} \dfrac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{otherwse.} \end{cases}$$

In other to see that the preceding makes mathematical sense, we must verify that: (i) $f(x) \geq 0$ for all x; and (ii) $\int_{-\infty}^{\infty} f = 1$. Part (i) is trivial; to see (ii) we change variables: If $x \geq 0$ then

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\, dx &= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-\lambda x}\, dx \\ &= \frac{1}{\Gamma(r)} \int_0^\infty z^{r-1} e^{-z}\, dz \qquad [z := \lambda x] \\ &= 1, \end{aligned}$$

since the final integral is $\Gamma(r)$, by definition.

Next let us compute EX:

$$EX = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x \cdot x^{r-1} e^{-\lambda x}\, dx = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^r e^{-\lambda x}\, dx.$$

Change variables $[z := \lambda x]$ to find that

$$EX = \frac{\lambda^r}{\Gamma(r)\lambda^{r+1}} \int_0^\infty z^r e^{-z}\, dz = \frac{\lambda^r}{\Gamma(r)\lambda^{r+1}} \Gamma(r+1) = \frac{r}{\lambda},$$

thanks to the duplication formula for gamma functions. Similarly,

$$E(X^2) = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r+1} e^{-\lambda x}\, dx = \frac{\lambda^r}{\Gamma(r)\lambda^{r+2}} \Gamma(r+2) = \frac{r(r+1)}{\lambda^2}.$$

Therefore,

$$\text{Var}X = \frac{r(r+1)}{\lambda^2} - \left(\frac{r}{\lambda}\right)^2 = \frac{r}{\lambda^2} \qquad \Rightarrow \qquad \text{SD}(X) = \frac{\sqrt{r}}{\lambda}.$$