Solutions to Homework 7

Math 5010-1, Summer 2010

July 12, 2010

p. 275–277, #3. We will need the antiderivative of $x(1-x)$ several times; therefore let us compute that first:

$$\int x(1-x)\,dx = \int x\,dx - \int x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}.$$

(a) Need to choose c so that $\int_{-\infty}^{\infty} f = 1$. That is,

$$1 = c\int_0^1 x(1-x)\,dx = c\left(\left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^3}{3}\right|_0^1\right) = \frac{c}{6}.$$

Therefore, $c = 6$.

(b) The answer is "one half, by symmetry." Alternatively,

$$\begin{aligned} P\{X \le 1/2\} &= \int_0^{1/2} 6x(1-x)\,dx = 6\left(\left.\frac{x^2}{2}\right|_0^{1/2} - \left.\frac{x^3}{3}\right|_0^{1/2}\right) \\ &= \frac{1}{2}. \end{aligned}$$

(c) We compute

$$\begin{aligned} P\{X \le 1/3\} &= \int_0^{1/3} 6x(1-x)\,dx = 6\left(\left.\frac{x^2}{2}\right|_0^{1/3} - \left.\frac{x^3}{3}\right|_0^{1/3}\right) \\ &= \frac{7}{27}. \end{aligned}$$

(d) We can compute directly by integration, or observe that

$$P\{1/3 < X < 1/2\} = P\{X < 1/2\} - P\{X < 1/3\} = \frac{1}{2} - \frac{7}{27} = \frac{13}{27}.$$

(e) As for the mean,

$$EX = 6\int_0^1 x^2(1-x)\,dx = 6\left(\int_0^1 x^2\,dx - \int_0^1 x^3\,dx\right) = \frac{1}{2}.$$

Next we compute the second moment of X:

$$E(X^2) = 6\int_0^1 x^3(1-x)\,dx = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{10}.$$

Therefore, $\mathrm{Var}X = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

p. 275–277, #6. (a) We know that

$$0.3333 \approx \frac{1}{3} = P\{X \le 0\} = \Phi\left(-\frac{\mu}{\sigma}\right) = 1 - \Phi\left(\frac{\mu}{\sigma}\right).$$

Equivalently, $\Phi(\mu/\sigma) \approx 0.6667$. This and the normal table together tell us that

$$\frac{\mu}{\sigma} \approx 0.43.$$

Similarly,

$$0.6667 \approx \frac{2}{3} = P\{X \le 1\} = \Phi\left(\frac{1-\mu}{\sigma}\right).$$

Therefore,

$$\frac{1-\mu}{\sigma} \approx 0.43.$$

In other words,

$$0.43 \approx \frac{1-\mu}{\sigma} \approx \frac{1}{\sigma} - 0.43$$

Therefore, $\sigma \approx 1/(2\times 0.43) \approx 1.162$ and $\mu \approx 0.43\sigma \approx 0.43 \times 1.162 \approx 0.499$.

(b) If instead $P\{X \le 1\} = \frac{3}{4}$, then

$$0.75 = P\{X \le 1\} = \Phi\left(\frac{1-\mu}{\sigma}\right) \quad\Rightarrow\quad \frac{1-\mu}{\sigma} \approx 0.67.$$

Therefore,

$$0.67 \approx \frac{1}{\sigma} - 0.43 \quad\Rightarrow\quad \sigma \approx \frac{1}{1.1} = \frac{10}{11}.$$

And therefore, $\mu \approx 0.43\sigma = \frac{4.3}{11} \approx 0.39$.

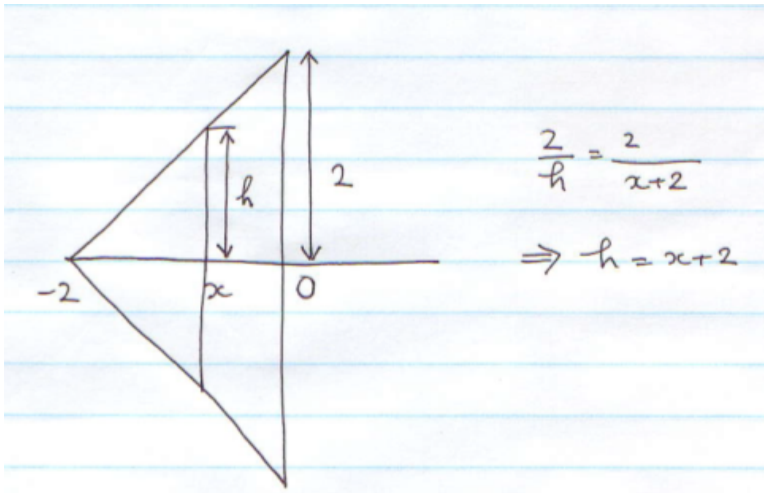

Figure 1: Problem 12(a), pp. 275–277

p. 275–277, #12. The solution is obtained as follows. First compute the following for all x:

$$P\{X \le x\} = \int_{-\infty}^{x} f.$$

Then use the fundamental theorem of calculus to deduce that

$$f(x) = \frac{d}{dx} P\{X \le x\}.$$

(a) According to the picture $f(x) = 0$ if $x \le -2$ or $x \ge 2$. If $-2 < x < 2$, then $P\{X \le x\}$ is the area to the left of x on the horizontal axis divided by the total area which is 8. [For instance, if $x = 0$, then this area is 1/2.] We compute the area depending on whether or not $-2 < x < 0$:

i. If $x < 0$, then the area to the left of x is the area of a triangle of base $2(x+2)$ and height $x+2$; see Figure 1. The area of that triangle is $\frac{1}{2} \times 2(x+2) \times (x+2) = (x+2)^2$. Therefore, whenever $-2 < x < 0$,

$$P\{X \le x\} = \frac{(x+2)^2}{8} \quad \Rightarrow \quad f(x) = \frac{2(x+2)}{8} = \frac{x+2}{4}.$$

And by similar [symmetric] arguments, if $2 > x > 0$ then

$$f(x) = \frac{-x+2}{4}.$$

The other pictures are handled similarly; you have to work with the shapes a bit more. Because the remaining ideas are those of elementary geometry [and not probability] I will leave them for you to mull over.

p. 293–295, #3. We know that if $X :=$ time to the next earthquake, then $P\{X > t\} = e^{-t}$ for all $t > 0$.

(a) $P\{X > 1\} = e^{-1}$.

(b) $P\{X > 1/2\} = e^{-1/2}$.

(c) $P\{X > 2\} = e^{-2}$.

(d) $P\{X > 10\} = e^{-10}$.

p. 293–295, #4. Let X be the lifetime of a randomly-selected component. We know that $EX = 10$, therefore, X is exponentially distributed with parameter $\lambda = 1/10$. That is, $P\{X > t\} = e^{-t/10}$.

(a) $P\{X > 20\} = e^{-20/10} = e^{-2}$.

(b) We wish to find a number m such that

$$\frac{1}{2} = P\{X > m\} = e^{-m/10} \quad \Rightarrow \quad m = 10 \ln 2.$$

(c) The SD of an exponential with parameter λ is λ; therefore, $\mathrm{SD}(X) = 10$.

(d) If $X_1, \ldots, X_{100}$ denote the respective lifetimes, then by the central limit theorem,

$$P\left\{\frac{X_1 + \cdots + X_{100}}{100} > 11\right\} \approx 1 - \Phi\left(\frac{11-\mu}{\sigma}\right) = 1 - \Phi(0.1),$$

where $\mu = \sigma = 10$. This probability is very close to $1 - 0.54 = 0.46$.

(e) We need to know that the sum of the two exponentials is a gamma. But we skip this for now.

p. 293–295, #7. We know that $P\{T \leq t\} = 1 - e^{-\lambda t}$ for all $t > 0$. Therefore,

$$p = 1 - e^{-\lambda t_p} \quad \Rightarrow \quad t_p = \frac{1}{\lambda} \ln\left(\frac{1}{1-p}\right).$$