Solutions to Homework 4

Math 5010-1, Summer 2010

June 17, 2010

p. 132–137, #4. I will assume that the tosses are independent from one another. Let H_k denote the event that the kth toss results in haeds [and $T_k := H_k^c :=$ the kth toss is tails]. We are asked to compute

$$P(H_{10}\,|\,\text{at least 9 tails}) = \frac{P(T_1 \cap \cdots \cap T_9 \cap H_{10})}{P(T_1 \cap \cdots \cap T_9)} = \frac{(1/2)^{10}}{\mathcal{P}},$$

where $\mathcal{P} :=$ the probability that there are at least 9 tails. Let $X :=$ the total number of tails tossed. Then,

$$\begin{aligned}
\mathcal{P} &= P\{X \geq 9\} \\
&= P\{X = 9\} + P\{X = 10\} \\
&= \binom{10}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{10} = \frac{10}{2^{10}} + \frac{1}{2^{10}} \\
&= \frac{11}{2^{10}}.
\end{aligned}$$

Therefore,

$$P(H_{10}\,|\,\text{at least 9 tails}) = \frac{2^{-10}}{11 \times 2^{-10}} = \frac{1}{11}.$$

p. 132–137, #7. Let "success" denote "heads." Then we want

$$P\{\text{between } 500,000 - k \text{ and } 500,000 + k \text{ successes}\} \approx 0.96.$$

We are going to use the normal approximation; but first note that $n = 10^6$ [number of trials] and $p = 1/2$ [probab. of success per trial]. Therefore, $\mu = np = \frac{1}{2} \times 10^6 = 500,000$ and $\sigma = \sqrt{npq} = \sqrt{10^6 \times (1/2)^2} = 500$. By the central limit theorem,

$$\begin{aligned}
&P\{\text{between } 500,000 - k \text{ and } 500,000 + k \text{ successes}\} \\
&\approx \Phi\left(\frac{500,000 + k - \mu}{\sigma}\right) - \Phi\left(\frac{500,000 - k - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{k}{500}\right) - \Phi\left(-\frac{k}{500}\right).
\end{aligned}$$

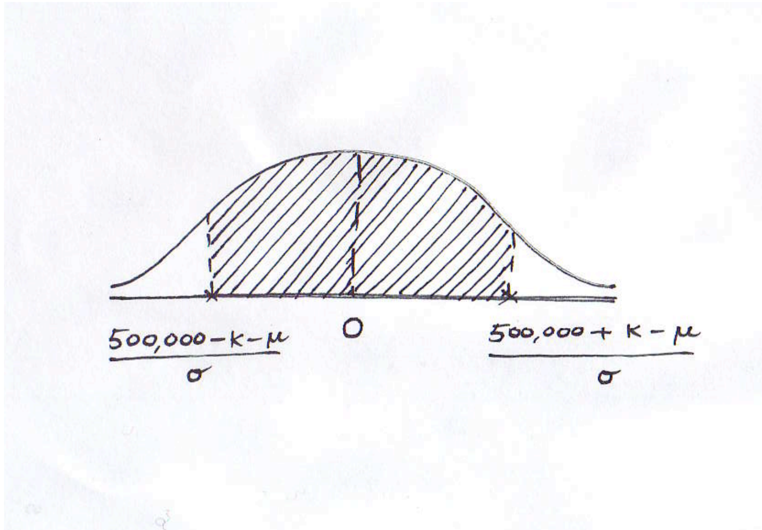

Figure 1: The probability area for problem 7, p. 132

Figure 1 [below] shows a picture of this probability approximation. We want the shaded region's area to be approximately 0.96; this means that the area to the left of $k/500$ has to be ≈ 0.98. Therefore, we use the normal table to find that

$$\frac{k}{500} \approx 2.06 \quad \Rightarrow \quad k \approx 500 \times 2.06 = 1030.$$

p. 132–137, #13. First of all, let us compute the probability p that a given packet contains 3 or more seeds that do not germinate. We can note that

$$\begin{aligned} p &= P\{47 \text{ or more germinate}\} \\ &= 1 - [P\{\text{all germinate}\} + P\{49 \text{ germinate}\} + P\{48 \text{ germinate}\}] \\ &= 1 - \left[0.99^{50} + \binom{50}{1}(0.99^{49} \times 0.01) + \binom{50}{2}(0.99^{48} \times 0.01^2)\right] \\ &\approx 0.014. \end{aligned}$$

[These are binomial probabilities.] Call it a "success" every time a packet has at least 3 nongerminating seeds. We are asked to compute that in binomial trials with $n = 4,000$ and

$p \approx 0.014$ [so $\mu = np \approx 56$ and $\sigma = \sqrt{npq} \approx 7.4$],

$$P\{\text{at least 40 successes}\} \approx 1 - \Phi\left(\frac{40-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{40-56}{7.4}\right) = 1 - \Phi(-2.07) = \Phi(2.07) \approx 0.98.$$

p. 158–161, #3. The range [or the set of possible values] of S is the collection of all integers between 2 and 12. The distribution is as follows:

Possible value	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

p. 158–161, #6. (a)

X \ Y	0	1	2
0	$(1/2)^3 = \frac{1}{8}$	$(1/2)^3 = \frac{1}{8}$	0
1	$(1/2)^3 = \frac{1}{8}$	$2 \times (1/2)^3 = \frac{2}{8}$	$(1/2)^3 = \frac{1}{8}$
2	0	$(1/2)^3 = \frac{1}{8}$	$(1/2)^3 = \frac{1}{8}$

(b) No. For instance, $P\{X = 2\,, Y = 0\} = 0$, whereas $P\{X = 2\} = (1/8) + (1/8) = \frac{1}{4}$, and $P\{Y = 0\} = \frac{1}{4}$, therefore, $P\{X = 2\} \times P\{Y = 0\} = \frac{1}{16} \neq 0$.

(c) The possible values of $X + Y$ are 0, 1, 2, 3, and 4. For instance, $P\{X + Y = 2\} = P\{X = 0\,, Y = 2\} + P\{X = 2\,, Y = 0\} + P\{X = 1\,, Y = 1\} = \frac{2}{8}$. In general, we have

Possible value	Probability
0	$\frac{1}{8}$
1	$\frac{1}{8} + \frac{1}{8} = \frac{2}{8}$
2	$\frac{2}{8}$
3	$\frac{1}{8} + \frac{1}{8} = \frac{2}{8}$
4	$\frac{1}{8}$

p. 158–161, #10. (a) S_n is the total number of successes in n Bernoulli trials. Therefore, S_n has the binomial distribution with parameters n and p.

(b) T_m is the total number of successes in m Bernoulli trials. Therefore, T_m has the binomial distribution with parameters m and p.

(c) S_n+T_m is the total number of successes in $n+m$ Bernoulli trials. Therefore, S_n+T_m has the binomial distribution with parameters $n+m$ and p.

(d) Yes because they are counted based on independent trials. In other words, we have discovered the following: *If X is binomial $(n\,,p)$ and Y is an independent binomial $(m\,,p)$, then $X+Y$ is binomial $(n+m\,,p)$.*

p. 182–184, #6. Let A_j denote the event that the jth card is a spade. Then $P(A_j)=\frac{13}{52}=\frac{1}{4}$. Also, $I_{A_1}+\cdots+I_{A_7}$ denotes the total number of cards that are spades. Therefore, $X=I_{A_1}+\cdots+I_{A_7}$, and by the addition rule for expectations,

$$E(X)=E\left(I_{A_1}+\cdots+I_{A_7}\right)=P(A_1)+\cdots+P(A_7)=\frac{7}{4}.$$

p. 182–184, #14. Let A_j denote the event that the button for floor j has been pressed. The total number X of visited floors is the random variable $I_{A_1}+\cdots+I_{A_{10}}$. Therefore,

$$E(X)=P(A_1)+\cdots+P(A_{10}).$$

Now every $P(A_j)$ is the probability that at least one of the 12 people in the elevator presses the button to floor j. Therefore,

$$P(A_j)=1-\left(\frac{9}{10}\right)^{12}\approx 0.717570463519.$$

This yields

$$E(X)=10\left\{1-\left(\frac{9}{10}\right)^{12}\right\}\approx 7.17570463519.$$

p. 182–184, #20. First of all,

$$\mu_1=E(X)=P\{X=1\}+2P\{X=2\}.$$

Also,

$$\mu_2=P\{X=1\}+4P\{X=2\}.$$

This yields two linear equations in to unknown, which we solve next: Subtract the two equations to find that

$$\mu_2 - \mu_1 = 2P\{X = 2\} \qquad \Rightarrow \qquad P\{X = 2\} = \frac{\mu_2 - \mu_1}{2}.$$

Then plug the latter into the equation for μ_1 [say] to find that

$$\mu_1 = P\{X = 1\} + 2\left(\frac{\mu_2 - \mu_1}{2}\right) = P\{X = 1\} + \mu_2 - \mu_1.$$

Therefore,

$$P\{X = 1\} = 2\mu_1 - \mu_2.$$

Finally,

$$\begin{aligned} P\{X = 0\} &= 1 - [P\{X = 1\} + P\{X = 2\}] \\ &= 1 - \left[2\mu_1 - \mu_2 + \left(\frac{\mu_2 - \mu_1}{2}\right)\right] \\ &= 1 - \left[\frac{3\mu_1 - \mu_2}{2}\right]. \end{aligned}$$