Solutions to Homework 2

Math 5010-1, Summer 2010

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p. 53–55, #2. Let W_i denote the event that the ith ball drawn is white. And let B_i denote the same event, but for a black ball [that is, $B_i = W_i^c$].

(a) We know the following: $P(W_1) = \frac{4}{10} = \frac{2}{5}$; $P(B_1) = \frac{3}{5}$; $P(W_2 \mid W_1) = \frac{7}{13}$; and $P(W_2 \mid B_1) = \frac{4}{13}$. Therefore, the law of total probability tells us that

$$\begin{aligned} P(W_2) &= P(W_2 \mid W_1)P(W_1) + P(W_2 \mid B_1)P(B_1) \\ &= \left(\frac{7}{13} \times \frac{2}{5}\right) + \left(\frac{4}{13} \times \frac{3}{5}\right) = \frac{26}{65} \qquad \left[= \frac{4}{10}\right]. \end{aligned}$$

(b) We apply Bayes' rule to find that

$$\begin{aligned} P(W_1 \mid W_2) &= \frac{P(W_2 \mid W_1)P(W_1)}{P(W_2)} \\ &= \frac{\frac{7}{13} \times \frac{4}{10}}{\frac{26}{65}}. \end{aligned}$$

Of course, $P(B_1 \mid W_2) = 1 - P(W_1 \mid W_2)$. Or you can compute $P(B_1 \mid W_2)$ directly [check!]

(c) Now we know the following: $P(W_1) = \frac{w}{w+b}$; $P(B_1) = \frac{b}{w+b}$; $P(W_2 \mid W_1) = \frac{w+d}{w+b+d}$; $P(W_2 \mid B_1) = \frac{w}{w+b+d}$. Therefore,

$$\begin{aligned} P(W_2) &= P(W_2 \mid W_1)P(W_1) + P(W_2 \mid B_1)P(B_1) \\ &= \left(\frac{w+d}{w+b+d} \times \frac{w}{w+b}\right) + \left(\frac{w}{w+b+d} \times \frac{b}{w+b}\right). \end{aligned}$$

We may factor $w/(w+b+d)(w+b)$ from the sum to find that

$$P(W_2) = \frac{w}{(w+b+d)(w+b)} \times (w+d+b) = \frac{w}{w+b}.$$

p. 53–55, #5. Let D denote the event that the randomly-selected person has the disease. Also let DD denote the event that the diagnosis is that [the very same] person has the disease. We know the following facts: $P(D) = 0.01$; $P(DD \mid D^c) = 0.05$ and $P(DD \mid D) = 0.8$.

(a) We may apply the law of total probability to find that

$$\begin{aligned} P(DD) &= P(DD \mid D)P(D) + P(DD \mid D^c)P(D^c) \\ &= (0.8 \times 0.01) + (0.05 \times 0.99) = 0.0575. \end{aligned}$$

(b) We are asked to computed $P(D \cap DD^c)$ [note the correction!]. Apply Bayes' rule to find that

$$P(DD^c \cap D) = P(DD^c \mid D)P(D) = (1 - 0.8) \times 0.01 = 0.02.$$

(c) We are asked to compute

$$\begin{aligned} P(DD^c \cap D^c) &= P(DD^c \mid D^c)P(D^c) = (1 - 0.05) \times (1 - 0.1) \\ &= 0.855. \end{aligned}$$

(d) Once again, by Bayes' rule.

$$\begin{aligned} P(D \mid DD) &= \frac{P(DD \mid D)P(D)}{P(DD \mid D)P(D) + P(DD \mid D^c)P(D^c)} \\ &= \frac{0.8 \times 0.01}{(0.8 \times 0.01) + (0.05 \times 0.99)} \approx 0.139. \end{aligned}$$

(e) For most diagnoses [this is a long-run statement, why?], the test works fairly well in the following sense: Among most randomly-selected people [in a long-run sense; sort this out], 1% have this disease. So roughly 1% of the time we can sample a person with the disease. But among most people who have had a positive diagnosis, 13.9% have the disease!

pp. 70–71, #1. This is the corrected version. Assume all births are independent from one another. If there are N people and $N \leq 12$, then [by independence] the chances are

$$\underbrace{\frac{12}{12} \times \frac{11}{12} \times \cdots \times \frac{12 - N + 1}{12}}_{\text{there are } N \text{ terms in this product; sort this out}} = \frac{11}{12} \times \cdots \times \frac{12 - N + 1}{12}$$

that no two people have the same sign. If $N \geq 13$, then the chances are zero and there is nothing to compute. Therefore, we want the probability that there is at least two shared birthdays to be 50% or greater; that is

$$1 - \left[\frac{11}{12} \times \cdots \times \frac{12-N+1}{12}\right] \geq 0.5.$$

Equivalently, we want

$$P(N) := \frac{11}{12} \times \cdots \times \frac{12-N+1}{12} \leq 0.5 \qquad [2 \leq N \leq 12].$$

Therefore, we start computing $P(2)$, $P(3)$, etc. until we arrive at 0.5 or less:

$P(2)$	$\frac{11}{12} = 0.91\bar{6}$
$P(3)$	$P(2) \times \frac{10}{12} = 0.73\bar{8}\bar{3}$
$P(4)$	$P(3) \times \frac{9}{12} \approx 0.5729$
$P(5)$	$P(4) \times \frac{8}{12} = 0.3819\bar{4}$

Therefore, we need at least 5 people.

pp. 70–71, #3. We know that $P(H) = \frac{2}{3}$ for every toss. Let N_j denote the event that there were j heads tossed in the 3 tosses.

(a) The probability of tossing at least one head is

$$1 - P(N_0) = 1 - P(T_1T_2T_3) = 1 - \left(\frac{1}{3}\right)^3 = \frac{26}{27}.$$

Also, the probability of tossing at least two heads $[N_2 \cup N_3]$ is

$$\begin{aligned}
&P(N_2) + P(N_3) \\
&= [P(H_1H_2T_3) + P(H_1T_2H_3) + P(T_1H_2H_3)] + P(H_1H_2H_3) \\
&= \left[3 \times \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)\right] + \left(\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 \\
&= \left(\frac{2}{3}\right)^2 \left\{1 + \frac{2}{3}\right\} = \frac{4}{9} \times \frac{5}{3} = \frac{20}{27}.
\end{aligned}$$

Therefore, the answer that we seek is

$$P(N_2 \cup N_3 \mid N_0^c) = \frac{\frac{20}{27}}{\frac{26}{27}} = \frac{20}{26}.$$

(b) We are asked to find $P(N_1 \mid N_0^c) = 1 - \frac{20}{26} = \frac{3}{13}$. [Note the correction!]

pp. 70–71, #5. Assumed that all birthdays are independent, and every year has 365 equally-likely days. Also, $n \leq 365$; otherwise the problem is elementary.

(a) The chances are

$$1 - \left(\frac{364}{365}\right)^{n-1}.$$

(b) We set $1 - (364/365)^{n-1} \geq 0.5$, and solve for n. Equivalently,

$$\left(\frac{364}{365}\right)^{n-1} \leq 0.5 \qquad \Leftrightarrow \qquad (n-1)\ln\left(\frac{364}{365}\right) \leq \ln(0.5),$$

which means that

$$n-1 \geq \frac{\ln(0.5)}{\ln(364/365)} \quad \Leftrightarrow \quad n \geq 1+\frac{\ln(0.5)}{\ln(364/365)} \approx 253.652.$$

So we have to choose $n \geq 254$.

pp. 70–71, #7. We apply independence several times:

(a) $p_1p_3+p_2p_3-p_1p_2p_3$ [either the top works or the bottom or both; but this is not a disjoint union, hence the last term ...]

(b) The probability that the 1-2-3 component *fails* is $1 - (p_1p_3 + p_2p_3 - p_1p_2p_3)$, by part (a). The probability that both 4 and 1-2-3 fail is therefore,

$$p_4\left\{1 - (p_1p_3 + p_2p_3 - p_1p_2p_3)\right\},$$

and it follows from the preceding that the probability that at least one of the two [i.e., "4" and/or "1-2-3"] work is

$$1 - p_4\left\{1 - (p_1p_3 + p_2p_3 - p_1p_2p_3)\right\}.$$

I can't think of a good reason why this formula should be simplified. Therefore, I will not simplify it [but you can, if you want].

pp. 70–71, #8. Note that $P(B_{ij}) = \frac{365}{365} \times \frac{1}{365} = \frac{1}{365}$.

(a) $B_{12} \cap B_{23}$ is the event that persons 1, 2, and 3 have a common birthday. Therefore,

$$P(B_{12} \cap B_{23}) = \frac{365}{365} \times \frac{1}{365} \times \frac{1}{365} = \frac{1}{365^2}.$$

This is equal to $P(B_{12})P(B_{23})$; therefore, B_{12} and B_{23} are indeed independent.

(b) Because $B_{12} \cap B_{23} \cap B_{31} = B_{12} \cap B_{23}$, it follows that $P(B_{12} \cap B_{23} \cap B_{31}) = \frac{1}{365^2}$, whereas $P(B_{12})P(B_{23})P(B_{31}) = \frac{1}{365^3}$. Therefore, the three events are *not* independent.

(c) Yes. B_{12} and B_{23} were shown to be independent in (1). We can carry out a similar computation for the remaining pairs, (B_{12}, B_{31}) and (B_{23}, B_{31}).