Solutions to Homework 1

Math 5010-1, Summer 2010

May 26, 2010

p. 10, #7. Let $p(i\,,j)$ denote the probability that we roll i dots on the first die and j dots on the second. Note that $p(i\,,j) = 1/36$ for every fixed choice of $i, j = 1, \ldots, 6$.

(a) We want

$$p(1\,,1) + p(1\,,2) + p(2\,,1) + p(2\,,2) = \frac{4}{36} = \frac{1}{9}.$$

(b) We want

$$\underbrace{\text{The above expression}}_{=4/36} + p(1\,,3) + p(2\,,3) + p(3\,,3) + p(3\,,1) + p(3\,,2)$$
$$= \frac{4}{36} + \frac{5}{36} = \frac{9}{36} = \frac{1}{4}.$$

(c) We want

$$p(1\,,3) + p(2\,,3) + p(3\,,3) + p(3\,,1) + p(3\,,2) = \frac{5}{36}.$$

(d) Let $P(x)$ denote the probability that the maximum is exactly equal to x. Then we work in steps:

$$P(1) = p(1\,,1) = \frac{1}{36};$$

$$P(2) = p(1\,,2) + p(2\,,1) + p(2\,,2) = \frac{3}{36} \quad \left[= \frac{1}{8}\right];$$

$$P(3) = \frac{5}{36}; \qquad \text{[as before]}$$

$$P(4) = p(1\,,4) + \cdots + p(4\,,4) + p(4\,,3) + \cdots + p(4\,,1) = \frac{7}{36};$$

$$P(5) = p(1,5) + \cdots + p(5,5) + p(5,4) + \cdots + p(5,1) = \frac{9}{36} \quad \left[= \frac{1}{4}\right];$$

$$P(6) = p(1,6) + \cdots + p(6,6) + p(6,5) + \cdots + p(6,1) = \frac{11}{36}.$$

We are asked to also compute $Q(x) :=$ probability that the max is x or less. Once again we work in steps:

$$Q(1) = P(1) = \frac{1}{36};$$

$$Q(2) = P(1) + P(2) = \frac{1}{36} + \frac{3}{36} = \frac{4}{36};$$

$$Q(3) = P(1) + P(2) + P(3) = Q(2) + P(3) = \frac{4}{36} + \frac{5}{36} = \frac{9}{36};$$

$$Q(4) = Q(3) + P(4) = \frac{9}{36} + \frac{7}{36} = \frac{16}{36};$$

$$Q(5) = Q(4) + P(5) = \frac{16}{36} + \frac{9}{36} = \frac{25}{36};$$

$$Q(6) = Q(5) + P(6) = \frac{25}{36} + \frac{11}{36} = 1.$$

(e) 1, because the maximum of the two is one, and exactly one, of $1, \ldots, 6$. Alternatively, $Q(6) = 1$. Yet another way to see this is

$$P(1) + \cdots + P(6) = \frac{1}{36} + \frac{3}{36} + \frac{5}{36} + \frac{7}{36} + \frac{9}{36} + \frac{11}{36} = 1.$$

p. 30–31, #2. (a) $(A \cap B^c) \cup (A^c \cap B)$; alternatively, $(A \setminus B) \cup (B \setminus A)$.

(b) $(A \cup B \cup C)^c$.

(c) i. Exactly one of A, B, or C:

$$(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C).$$

ii. Exactly two of A, B, or C:

$$(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C).$$

iii. Exactly three of A, B, or [and] C:

$$A \cap B \cap C.$$

p. 30–31, #6. i. That sentence has 10 words: One 1-letter word ["a"], two 2-letter words ["is" and "at"], three 4-letter words ["word," "from," and "this"], two 6-letter words ["picked" and "random"], and two 7-letter words ["Suppose" and "sentence"]. Therefore, the probabilities are distributed as follows:

Number of letters	**Probability**
1	0.1
2	0.2
4	0.3
6	0.2
7	0.2

ii. Similarly, six words have 1 vowel ["a," "word," "is," "at," "from," and "this"], two words have 2 vowels ["picked" and "random"], and two words have 3 vowels ["Suppose" and "sentence"] Therefore, the distribution of the vowel counts is the following:

Number of vowels	**Probability**
1	0.6
2	0.2
3	0.2

p. 45–46, #3.

$$P(\text{rain tomorrow} \mid \text{rain today}) = \frac{P(\text{rain tomorrow and today})}{P(\text{rain today})} = \frac{3}{4}.$$

p. 45–46, #4. Let us call the events A and B, so that $P(A) = 0.1$ and $P(B) = 0.3$ [say].

(a) We are asked to compute $1 - P(A \cup B)$. Since that $P(A \cap B) = P(A) + P(B) - P(A \cap B)$, independent tells us that

$$P(A \cap B) = 0.1 + 0.3 - (0.1 \times 0.3) = 0.37.$$

Therefore, the answer is $1 - 0.37 = 0.63$.

(b) $P(A \cup B) = 0.37$; see the previous part.

(c) The answer is $P(A \cap B^c) + P(A^c \cap B)$. By independence,

$$P(A \cap B^c) = P(A)P(B^c) = 0.1 \times 0.7 = 0.07,$$

and

$$P(A^c \cap B) = P(A^c)P(B) = 0.9 \times 0.3 = 0.27.$$

Therefore, the answer is $0.07 + 0.27 = 0.34$.

p. 45–46, #8. This problem is worded loosely. There are many ways to interpret it. Here is one: You select a card at random with the given probabilities: respectively, 0.3 for double-white; 0.5 for black/white; and 0.2 for double-black cards. If you pick a card, then you put it face down on the black side, if it has one. We know that

$$P(WW) = 0.3, \; P(BW) = 0.5, \text{ and } P(BB) = 0.2,$$

and want $P(BW \mid B?)$ [the notation ought to be clear here]. We can apply the conditional-probability formula:

$$P(BW \mid B?) = \frac{P(BW)}{P(B?)} = \frac{0.5}{P(BW) + P(BB)} = \frac{0.5}{0.5 + 0.2} = \frac{5}{7}.$$

Now suppose we change the interpretation of the problem as follows: All three cards are sided [side 1 and side 2]. After you pick a card at random, you toss independently a coin. If that coin lands on heads then we choose side one; else, we select side 2. Then, as above,

$$P(\text{black and white} \mid \text{black side showing}) \\ = \frac{P(\text{black and white \& black side showing})}{P(\text{black side showing})}.$$

But now the numerator is

$$P(\text{black side showing} \mid \text{black and white}) \times P(\text{black and white}) = \frac{1}{4}.$$

And the denominator is

$$P(\text{black side showing} \mid \text{black and white}) \times P(\text{black and white}) \\ + P(\text{black side showing} \mid \text{double black}) \times P(\text{double black}) \\ + P(\text{black side showing} \mid \text{double white}) \times P(\text{double white}) \\ = \frac{1}{4} + 0.2 + 0 = 0.45.$$

p. 45–46, #12. We apply the difference rule; viz.,

$$P(F \mid G^c) = \frac{P(F \cap G^c)}{P(G^c)} = \frac{P(F) - P(FG)}{1 - P(G)}.$$