Math 5010
Final Exam Practice Sheet

1. An urn contains 12 balls; 10 are yellow and the rest are green. Excepting their color, the balls are identical. We sample 4 balls at random—without replacement—from this urn. Let X denote the number of green balls sample.

   (a) What is the distribution of X?

   (b) Compute $E(X)$ and $\text{Var}(X)$.

   (c) What is the $P\{X \le 1\}$?

2. Let X be uniformly distributed on the interval $(1\,,2)$. Find the number a that minimizes the function $Q(a) := E(|X - a|)$.
(Hint: First compute $Q(a)$ for all real numbers a, and then consider setting $Q'(a) = 0$ to find the minimum.)

3. Suppose X is distributed uniformly on the interval $(-2\,,3)$; that is, is density function is

$$f(a) = \begin{cases} C & \text{if } -2 < a < 3, \\ 0 & \text{otherwise,} \end{cases}$$

   for some constant C.

   (a) Compute C.

   (b) Compute $E(X)$ and $\text{Var}(X)$.

   (c) Compute $F(a) = P\{X \le a\}$ for every real number a.

4. Suppose X and Y are independent random variables with common density function

$$f_X(a) = f_Y(a) = \begin{cases} 1 & \text{if } 0 \le a \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Compute:

(a) The density function for the random variable $D := |X - Y|$.

(b) $P\{|X - Y| \leq 1/2\}$.

(c) $E(|X - Y|)$.

5. Suppose X is a standard normal random variable; recall that its density function is

$$f(a) = \frac{e^{-a^2/2}}{\sqrt{2\pi}} \qquad \text{for } -\infty < a < \infty.$$

Compute the density function for $Y := X^2$.

6. Let (X, Y) be jointly distributed, uniformly on the triangle whose vertices are $(-1, 0)$, $(0, 1)$, and $(1, 0)$. Compute the following for every x between -1 and 1:

(a) The conditional density function of Y, given $X = x$;

(b) $E(Y \mid X = x)$.

7. Suppose (X, Y) are jointly distributed with density function

$$f(x, y) = \begin{cases} Cxy & \text{if } 0 \leq x < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where C is some constant.

(a) Compute the value of C.

(b) Compute $\mathrm{Var}(Y)$ and $\mathrm{Var}(Y \mid X = 1/2)$; why are they different?

8. Let (X, Y) be jointly distributed with the following distribution:

		Y = 0	Y = 1	Y = 2
X	0	1/8	1/8	0
	1	1/8	1/4	1/8
	2	0	1/8	1/8

(a) Compute the conditional distribution of X given that $X+Y = 3$.

(b) Compute $E(X \mid X + Y = 3)$ and $\mathrm{Var}(X \mid X + Y = 3)$.

9. Suppose X is a random variable with density function f, and assume that X has a finite mean μ and a finite variance σ^2. Find a number z that minimizes $S(z) := E([X - z]^2)$.
(Hint: Compute $S(z)$ for all real numbers z, and then consider setting $S'(z) = 0$ to find the minimum.)

10. Suppose (X, Y) is a jointly distributed random vector with a uniform distribution on the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Find the joint distribution of (U, V), where

$$U := \frac{X+Y}{\sqrt{2}}, \qquad V := \frac{X-Y}{\sqrt{2}}.$$

(This problem will not be covered.)

11. Suppose (X, Y) is jointly distributed with density function

$$f(x, y) = \begin{cases} x + y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the respective densities f_X and f_Y of X and Y. Are X and Y independent?

12. Suppose X and Y are independent continuous random variables with identical density functions (i.e., $f_X(a) = f_Y(a)$ for all a).
    (a) Prove that $P\{X < Y\} = 1/2$.
    (b) Construct an example of a random vector (X, Y) with a discrete joint distribution such that: (a) X and Y are independent; and (b) $P\{X < Y\} \neq 1/2$.

13. Compute $E(X^4)$, where X has the standard normal distribution; it might help to recall that the density function of X is

$$f(a) = \frac{e^{-a^2/2}}{\sqrt{2\pi}}.$$

[Hint: You need to first understand why $E(X^2) = 1$.]

14. Suppose X and Y both have a common geometric distribution with parameter p [between 0 and 1]. That is,

$$P\{X = k\} = P\{Y = k\} = (1-p)^{k-1}p \qquad \text{for } k = 1, 2, \dots .$$

What is the distribution of $X + Y$?