Examples of Conditional Prob's

Ex 1. [#12, p104]

$S = \{\text{survives the delivery}\}$ $\qquad C = \{\text{had C-section}\}$.

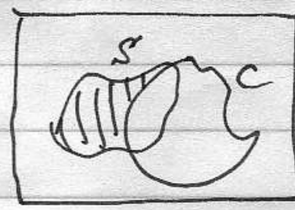
↑ not the sample space (!)

$P(S) = 0.98 \qquad P(C) = 0.15$

$P(S|C) = 0.96$.

Q: Find $P(S|C^c)$.

$$P(S|C^c) = \frac{P(S\cap C^c)}{P(C^c)}.$$



(a) $P(S\cap C^c) = P(S) - P(S\cap C)$

$= P(S) - P(S|C)P(C)$

$= 0.98 - 0.96 \times 0.15$

$= 0.836$

(b) $P(C^c) = 1 - 0.15 = 0.85 \Rightarrow P(S|C^c) = \dfrac{0.836}{0.85} = 0.983529$

$\simeq 0.984$.

Ex 2. [#16, p105]

$F = \{\text{female}\}$ $\qquad CS = \{\text{computer science major}\}$.

$P(F) = 0.52 \qquad P(CS) = 0.05 \qquad P(F\cap CS) = 0.02$

Q1 $\quad P(F|CS) = \dfrac{0.02}{0.05} = 0.4$

Q2 $\quad P(CS|F) = \dfrac{P(F\cap CS)}{P(F)} = \dfrac{0.02}{0.52} \simeq 0.0385$

Ex 3. [#33, p107] (a)

A fair coin, [redacted] and a 2-headed coin. Pick a coin at random and flip it. $P(\text{fair} \mid H) = ?$

$$P(\text{fair} \mid H) = \frac{P(\text{fair} \cap H)}{P(H)}$$

$$P(\text{fair} \cap H) = P(H \mid \text{fair})\, P(\text{fair}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

$$P(H) = P(H \mid \text{fair})\, P(\text{fair}) + P(H \mid \text{unfair})\, P(\text{unfair})$$
$$= \frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{2} = 3/4.$$

$$\Rightarrow P(\text{fair} \mid H) = \frac{1/4}{3/4} = \frac{1}{3}.$$

(Aside) ~~Ex~~

We've found the following:

Lemma:
$$P(A \mid B) = P(B \mid A)\, \frac{P(A)}{P(B)}.$$

Pf $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B \cap A)}{P(A)} \cdot \dfrac{P(A)}{P(B)} = P(B \mid A) \cdot \dfrac{P(A)}{P(B)}.$ #

Ex 4. [#33, p107] (b)

Suppose the gambler flips the same coin, flips it again, and again gets 'H'. $P(\text{fair} \mid H_1 H_2) = ?$

$$P(H_1 H_2 \mid \text{fair}) = \frac{1}{4}.$$

$P(\text{fair}) = 1/2$. So need $P(H_1 \cap H_2)$.

$$P(H_1 H_2) = P(H_1 H_2 \mid \text{fair})\, P(\text{fair}) + P(H_1 H_2 \mid \text{unfair})\, P(\text{unfair})$$
$$= \frac{1}{4} \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{5}{8}. \quad \therefore P(\text{fair} \mid H_1 H_2) = \frac{1}{4} \cdot \frac{1/2}{5/8} = \frac{1}{5}.$$

Ex. 5 [#33, P107] cont'd

Suppose the gambler flips the same coin but n times, and gets all H's. $P(\text{fair} \mid H_1 \cap H_2 \cap \cdots \cap H_n) = ?$

$$P(H_1 \cap \cdots \cap H_n \mid \text{fair}) = \frac{1}{2^n}.$$

$$P(\text{fair}) = \frac{1}{2}.$$

$$P(H_1 \cap \cdots \cap H_n) = P(H_1 \cdots H_n \mid \text{fair})\, P(\text{fair}) + P(H_1 \cdots H_n \mid \text{unfair})\, P(\text{unfair})$$

$$= \frac{1}{2^n} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}$$

$$= \frac{1}{2}\left[\frac{1}{2^n} + 1\right] = \frac{1}{2}\, \frac{2^n+1}{2^n} = \frac{2^n+1}{2^{n+1}}.$$

$$\therefore \quad P(\text{fair} \mid H_1 \cap \cdots \cap H_n) = \frac{1}{2^n} \times \frac{1/2}{2^n+1 \,/\, 2^{n+1}}$$

$$= \frac{1}{2^n}\, \frac{\frac{1}{2} \cdot 2^{n+1}}{2^n+1} = \frac{1}{2^n+1} \sim \frac{1}{2^n},$$

as $n \to \infty$.

Ex. 6 [#46, P108]

An insurance company has 3 classifications: good (g), average (a), and bad (b) risks.

$P(\text{accident} \mid g) = 0.05$

$P(\text{accident} \mid a) = 0.15$

$P(\text{accident} \mid b) = 0.3.$ $\quad P(g) = 0.2,\ P(a) = 0.5,\ P(b) = 0.3.$

Find $P(g \mid \text{no accident})$. $\quad$ Let $A := \{\text{accident}\}$.

$$P(g|A^c) = P(A^c|g)\,\frac{P(g)}{P(A^c)}$$

(i) $P(A^c|g) = 1 - P(A|g) = 0.95$ $\quad [P(A^c|a) = 0.85 \ ; \ P(A^c|b) = 0.7]$

(ii) $P(g) = 0.2$ $\quad [P(a) = 0.5 \ ; \ P(b) = 0.3]$

(iii) $P(A) = P(A|g)P(g) + P(A|a)P(a) + P(A|b)P(b) = 0.265.$

$\therefore \quad P(A^c) = 1 - 0.265 = 0.735.$

$\Rightarrow P(g|A^c) = 0.2585.$ What is going on?

$$\left[\begin{array}{l} P(a|A^c) = 0.578 \\ P(b|A^c) = 0.2857 \end{array}\right].$$