Random Vectors

The joint cdf:

$$F(a,b) = P\{X \le a, Y \le b\}.$$

The marginal cdf's:

$$F_X(a) = P\{X \le a\} = P\{X \le a, Y < \infty\} \overset{''}{=} \lim_{n\to\infty} P\{X \le a, Y \le n\}$$

$$= \lim_{n\to\infty} F(a,n) = F(a,\infty).$$

Also, $F_Y(b) = F(\infty, b)$.

The Discrete case $\quad p(x,y) = P\{X = x, Y = y\}$ the joint pmf

Marginal pmf's

$$p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y).$$

Ex – 3 Ⓦ, 4 Ⓑ — 4 draws w/o replacement.

X = # of Ⓦ drawn $\quad$ Y = # of Ⓑ's

$$p(0,4) = \binom{4}{4}\Big/\binom{7}{4} = \frac{1}{35} \qquad p(1,3) = \binom{3}{1}\binom{4}{3}\Big/\binom{7}{4} = \frac{12}{35}$$

$$p(2,2) = \binom{3}{2}\binom{4}{2}\Big/\binom{7}{4} = \frac{18}{35} \qquad p(3,1) = \binom{3}{3}\binom{4}{1}\Big/\binom{7}{4} = \frac{4}{35}$$

y \ x	0	1	2	3	p_Y
1	0	0	0	4/35	4/35
2	0	0	18/35	0	18/35
3	0	12/35	0	0	12/35
4	1/35	0	0	0	1/35
p_X:	$\frac{1}{35}$	$\frac{12}{35}$	$\frac{18}{35}$	$\frac{4}{35}$	1

The Continuous Case f = joint pdf :

- $P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy$.

- $F(a,b) = \int_{-\infty}^{b}\left[\int_{-\infty}^{a} f(x,y)\,dx\right]dy$

- $\frac{\partial^2 F}{\partial a \partial b}(a,b) = f(a,b)$.

- $P\{X \in A\} = P\{X \in A, Y \in \mathbb{R}\} = \int_{-\infty}^{\infty}\left[\int_A f(x,y)\,dx\right]dy$

$$= \int_A \underbrace{\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]}_{:= f_X(x) \,=\, \text{pdf of } X \text{ (marginal)}} dx$$

$\Rightarrow$

- $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

Ex. $f(x,y) = ?$ if (X,Y) is unif on $\bigcirc$ R (0,0) ?

$$f(x,y) = \begin{cases} C & x^2+y^2 \le R^2, \\ 0 & \text{else.} \end{cases}$$

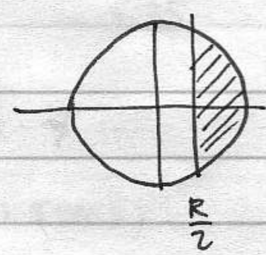
$$\iint_{\bigcirc} f(x,y)\,dx\,dy = C \cdot \text{Area}(\bigcirc) = C \cdot 4\pi R^2 = 1$$

$$\therefore \quad C = \frac{1}{4\pi R^2}.$$

$$\Rightarrow \quad f(x,y) = \begin{cases} \frac{1}{4\pi R^2} & x^2+y^2 \le R^2, \\ 0 & \text{else.} \end{cases}$$

- $P[X \le Y] = \frac{1}{2}$ (prove)

- $P[X \ge \frac{R}{2}] =$



$$= \int_0^R \int_{\frac{R}{2}}^R \frac{1}{\pi R^2}\, dx\, dy$$

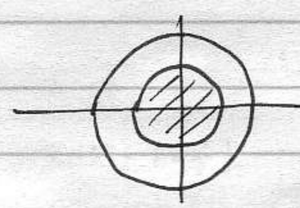
$$= \frac{R/2}{\pi R} = \frac{1}{2\pi}.$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\, dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad , x^2 \le R^2$$

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-x^2} & , -R \le x \le R \\ 0 & , \text{o/w} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2} & , -R \le y \le R \\ 0 & , \text{else.} \end{cases}$$

$P[\sqrt{X^2+Y^2} \le a] =$



$$\frac{\pi a^2}{\pi R^2} = \left(\frac{a}{R}\right)^2$$

$$0 \le a \le R$$

$\therefore$ if $Z = \sqrt{X^2+Y^2}$

$$f_Z(z) = \begin{cases} \frac{2z}{R^2} & 0 \le z \le R \\ 0 & \text{else} \end{cases} \qquad \text{eg.,} \quad EZ = \int_0^R \frac{2z^2}{R^2} dz = \frac{2}{3R^2} R^3 = \frac{2R}{3}.$$

<u>Independent r.v.'s</u> X and Y indept. if

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$$

So if X and Y are independent, their joint cdf is

$$F(a,b) = F_X(a)\, F_Y(b). \qquad \text{———} \circledast$$

Discrete: $\quad p(x,y) = \mathbb{P}(X=x)\, \mathbb{P}(Y=y) = p_X(x)\, p_Y(y)$

Cont. ? $\quad$ Differentiate $\circledast$

$$\frac{\partial}{\partial a} F(a,b) = f_X(a)\, F_Y(b)$$

$$f(a,b) = \frac{\partial^2}{\partial a \partial b} F(a,b) = f_X(a)\, f_Y(b).$$

<u>EX</u>. # of people entering a post office on a given day $\sim$ Pois(λ).
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Each person is male w/ prob p
female w/ prob $1-p$

X = # of men
Y = # of women. Find the joint mass fnc.

$$p(i,j) = \mathbb{P}(X=i, Y=j)$$

$$= \mathbb{P}(X=i, Y=j, X+Y=i+j)$$

$$= \mathbb{P}(X=i, Y=j \mid X+Y=i+j)\ \mathbb{P}(X+Y=i+j)$$

$$= \binom{i+j}{i} p^i (1-p)^j \times e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!}$$

$$= (\lambda p)^i \left[\lambda(1-p)\right]^j \frac{e^{-\lambda p}\, e^{-\lambda(1-p)}}{i!\, j!}$$

$$= \underbrace{\frac{e^{-\lambda p}(\lambda p)^i}{i!}}_{p_X(i)} \times \underbrace{\frac{e^{-\lambda(1-p)}(\lambda(1-p))^j}{j!}}_{p_Y(j)}$$

i.e. X and Y are independent Pois (λp), Pois $(\lambda(1-p))$, resp.

EX. $X, Y, Z \sim U(0,1)$ independent. Find $\mathbb{P}(X \geq YZ)$.

$$f_{(X,Y,Z)}(x,y,z) = \begin{cases} 1 & 0 \leq x,y,z \leq 1 \\ 0 & o/w \end{cases}$$

$$\mathbb{P}(X \geq YZ) = \iiint_{\substack{x \geq yz \\ 0 \leq x,y,z \leq 1}} 1 \, dx\,dy\,dz = \int_0^1 \int_0^1 \int_{yz}^1 dx\,dy\,dz$$

$$= \int_0^1 \int_0^1 (1 - yz)\,dy\,dz$$

$$= \int_0^1 \left(1 - \frac{1}{2}z\right) dz = \frac{3}{4}.$$

EX. X, Y indept. $X \sim \text{Bin}(n,p)$

$$Y = \begin{cases} 0 & 1/2 \\ 1 & 1/2 \end{cases}$$

$$\mathbb{P}(X > Y) = \sum_{\substack{x > y \\ x \in \{0 \ldots n\} \\ y = 0 \text{ or } 1}} p(x,y) = \sum_{x=1}^n p(x,0) + \sum_{x=2}^n p(x,1)$$

$$= \sum_{x=1}^n \frac{1}{2}\binom{n}{x} p^x (1-p)^{n-x} + \frac{1}{2}\sum_{x=2}^n \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{1}{2}\left[1 - \binom{n}{0} p^0 (1-p)^{n-0}\right] + \frac{1}{2}\left(1 - \binom{n}{0} p^0 (1-p)^{n-0} - \binom{n}{1} p^1 (1-p)^{n-1}\right]$$

$$= \frac{1}{2} - \frac{1}{2}(1-p)^n + \frac{1}{2} - \frac{1}{2}(1-p)^n - \frac{1}{2} np(1-p)^{n-1}$$

$$= 1 - (1-p)^n - \frac{1}{2} np(1-p)^{n-1}$$

$$= 1 - (1-p)^{n-1}\left[1 - p + \frac{np}{2}\right] = 1 - (1-p)^{n-1} \cdot \left[1 + \left(\frac{n}{2} - 1\right)p\right].$$