The Dist$^{\underline{n}}$ of a func of a rv.

Ex. $X \sim \text{Unif}(0,1)$. $Y = X^n$. What is $\mathcal{L}(Y)$?

Trick: Look at cdf's:

$$F_Y(y) = P\{Y \le y\} = P\{X^n \le y\}.$$

If $y \notin (0,1)$, then: $F_Y(y) = \begin{cases} 1, & y \ge 1, \\ 0, & y \le 0. \end{cases}$

So consider $0 \le y \le 1$. Then $F_Y(y) = P(X^n \le y) = P(X \le y^{1/n})$

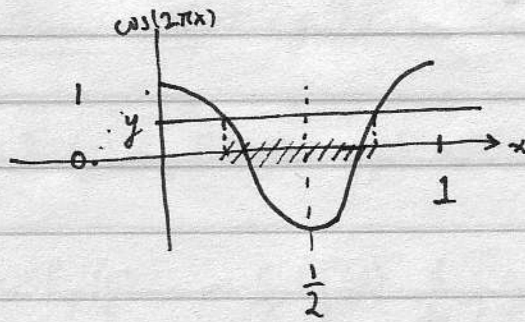
$$= \int_0^{y^{1/n}} dx = y^{1/n}.$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{n} y^{\frac{1}{n}-1}, & 0 \le y \le 1, \\ 0, & \text{o/w}. \end{cases}$$

(Harder) Ex. $X \sim \text{Unif}(0,1)$ $Y = \cos(2\pi X)$. $\mathcal{L}(Y) = ?$

$$F_Y(y) = P\left[\cos(2\pi X) \le y\right] = \begin{cases} 1 & y \ge 1, \\ 0 & y \le 1. \end{cases}$$

For $y \in (0,1)$, $\cos(2\pi X) \le y \longleftrightarrow$ $\cancel{\frac{1}{2\pi}|\cos^{-1}(y)| \le x \le -\left|\frac{1}{2\pi}\cos^{-1}(y)\right| + 1}$



$\cancel{F_Y(y) = 1 - \frac{1}{\pi}|\cos^{-1}(y)|}$

$$\cos(2\pi x) \le y \quad \longleftrightarrow \quad \frac{1}{2\pi}\cos^{-1}(y) \le x \le 1 - \frac{1}{2\pi}\cos^{-1}(y)$$

$$\therefore \quad F_Y(y) = P\left(\frac{1}{2\pi}\cos^{-1}(y) \le X \le 1 - \frac{1}{2\pi}\cos^{-1}(y)\right)$$

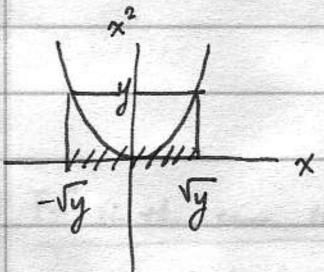
$$= 1 - \frac{1}{\pi}\cos^{-1}(y). \qquad 0 \le y \le 1.$$

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & 0 \le y \le 1 \\ 0 & o/w \end{cases}$$

Ex. $X \sim \text{Unif}(-1,1)$ $\qquad \left[f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & o/w \end{cases} \right]$

$Y = X^2$. $\quad \mathcal{L}(Y) = ?$ $\qquad F_Y(y) = 1 \quad y \ge 1 \qquad F_Y(y) = 0 \quad y \le 1.$

$$F_Y(y) = P(X^2 \le y) = P(X \le \sqrt{y}) - P(X \le -\sqrt{y})$$



$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

$$\therefore \quad f_Y(y) = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}} \qquad 0 \le y \le 1$$

$$= \frac{1}{2\sqrt{y}}.$$

EX. $X \sim Unif(-1,1)$, $Y = |X|$, $\mathcal{L}(Y) = ?$

$$F_Y(y) = \begin{cases} 1 & y \geq 1, \\ 0 & y \leq 0. \end{cases}$$

$y \in (0,1)$: $\quad F_Y(y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y)$

$$f_Y(y) = f_X(y) + f_X(-y) = 1.$$

$\therefore \quad Y \sim Unif(0,1).$

Theorem If X is a cont. r.v. w/ pdf f_X, and if g is strictly monotone & differentiable, continuously, then for $Y = g(X)$,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & , \text{ if } y = g(x) \text{ some } x, \\ 0 & , \text{ o/w.} \end{cases}$$

Pf in the case that $g \uparrow\uparrow$.

$$F_Y(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

$$\therefore \quad f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

Pf in the case that $g \downarrow\downarrow$.

$$F_Y(y) = P(g(X) \leq y) = P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$\therefore \quad f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

Ex. $X \sim f_X$, $X > 0$ a.s. $Y = X^n$.

$f_Y = ?$ $\quad Y = g(X) := X^n$. $\quad g^{-1}(y) = y^{1/n}$.

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{n} y^{1/n - 1} \qquad (>0 \quad \forall y>0).$$

$$f_Y(y) = \frac{1}{n} y^{\frac{1}{n}-1} f_X(y^{1/n}).$$

Ex. $Y = e^X = g(x)$

$$g^{-1}(y) = \ln y \qquad (y>0)$$

$$f_Y(y) = f_X(\ln y) \cdot \left| \frac{d}{dy} \ln y \right|$$

$$= f_X(\ln y) \cdot \frac{1}{y}.$$

Ex. In the previous example, if $X \sim \mathcal{E}xp(\lambda)$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0 & \text{else}. \end{cases} \qquad y = e^x$$

$$Y = e^X \Rightarrow \quad f_Y(y) = \begin{cases} \lambda e^{-\lambda \ln y} \quad (x \geq 0) & y \geq 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \lambda y^{-\lambda} & y \geq 1, \\ 0 & \text{else}. \end{cases}$$

"Pareto"