Ex $\alpha, \lambda > 0$ fixed. $X \sim \text{Gamma}(\alpha, \lambda)$ if its pdf is

$$f(x) = \frac{\lambda^\alpha e^{-\lambda x} x^{\alpha-1}}{\Gamma(\alpha)} \qquad (x>0) \quad , \text{ where}$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx. \qquad \left(f \geq 0, \ \int_{-\infty}^{\infty} f = 1 \ \text{ easy!}\right)$$

$$\left[\ \text{Gamma}\left(\tfrac{n}{2}, \tfrac{1}{2}\right) \triangleq \chi^2_n \ .\ \right]$$

<u>Aside</u> (Properties of the gamma function)

$$\Gamma(1) = 1$$

$$\alpha > 1: \ \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \qquad u = x^{\alpha-1} \quad v' = e^{-x}$$

$$= -x^{\alpha-1}e^{-x} \Big|_0^\infty + \int_0^\infty (\alpha-1) x^{\alpha-2} e^{-x} dx \qquad u' = (\alpha-1)x^{\alpha-2} \quad v = -e^{-x}$$

$$= (\alpha-1)\Gamma(\alpha-1).$$

$$\Rightarrow \quad \Gamma(n) = (n-1)!, \qquad n = 1, 2, \dots$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\, dx = \qquad (y = (2x)^{1/2})$$

$$= \int_0^\infty \frac{\sqrt{2}}{y} e^{-y^2/2}\, y\, dy \qquad x = y^2/2$$

$$dx = y\, dy$$

$$= \sqrt{2} \int_0^\infty e^{-y^2/2}\, dy$$

$$= \sqrt{2}\ \frac{1}{2} \sqrt{2\pi} = \sqrt{\pi}$$

$$\Rightarrow \ \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \quad \Gamma(5/2) = \tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi} \qquad \Gamma(7/2) = \tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\sqrt{\pi} \ \dots$$

$$E(X) = \int_0^\infty \frac{\lambda^\alpha x^\alpha e^{-\lambda x}}{\Gamma(\alpha)}\, dx = \frac{1}{\lambda}\int_0^\infty \frac{y^\alpha e^{-y}\, dy}{\Gamma(\alpha)} \qquad (y = \lambda x)$$

$$= \frac{1}{\lambda}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\alpha}{\lambda}.$$

$$E(X^2) = \int_0^\infty \frac{\lambda^\alpha x^{\alpha+1} e^{-\lambda x}}{\Gamma(\alpha)}\, dx = \frac{1}{\lambda^2 \Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-y}\, dy \qquad (y = \lambda x)$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\ \lambda^2} = \frac{(\alpha+1)\alpha}{\lambda^2}$$

$$\Rightarrow \quad \mathrm{Var}\, X = \frac{\alpha^2+\alpha}{\lambda^2} - \left(\frac{\alpha}{\lambda}\right)^2 = \alpha/\lambda^2 .$$

Ex- (Cauchy) $\quad f(x) = \dfrac{1}{\pi\{1+x^2\}}$

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\pi}\left[\arctan\infty - \arctan(-\infty)\right] = \frac{2\pi/2}{\pi} = 1 . \checkmark$$

$$E(X) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dx \quad \underline{\text{not defined}}!$$

$$\text{E.g.,} \int_{-n}^{n} \frac{x\, dx}{1+x^2} = 0 \quad \text{by symm.}$$

But

$$\int_{-n}^{2n} \frac{x\, dx}{1+x^2} = \int_n^{2n} \frac{x\, dx}{1+x^2} = \qquad (y = x^2)$$

$$= \frac{1}{2} \int_{\sqrt{n}}^{\sqrt{2n}} \frac{dy}{1+y} = \frac{1}{2} \ln\left(\frac{1+\sqrt{2n}}{\sqrt{n}}\right)$$

$$\to \frac{1}{2} \ln(\sqrt{2}) = \frac{1}{4} \ln 2 > 0 .$$

Ex – (Beta) $\quad a, b > 0 \quad X \sim \text{Beta}(a,b)$ if

$$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} \qquad 0 < x < 1, \quad \text{where}$$

B is the <u>beta function</u>, $\quad B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$.

<u>fact</u>: $\quad B(a,b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

$$E(X) = \frac{1}{B(a,b)} \int_0^1 x^{a}(1-x)^{b-1}\,dx$$

$$= \frac{B(a+1,b)}{B(a,b)} = \frac{\Gamma(a+1)\Gamma(b)\,/\,\Gamma(a+b+1)}{\Gamma(a)\Gamma(b)\,/\,\Gamma(a+b)}$$

$$= \frac{a}{a+b}.$$

$$E(X^2) = \frac{1}{B(a,b)} \int_0^1 x^{a+1}(1-x)^{b-1}\,dx = \frac{B(a+2,b)}{B(a,b)}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)} \div \frac{\Gamma(a+b)}{\Gamma(a+b+2)} = \frac{(a+1)\,a}{(a+b+1)(a+b)}.$$

$$\text{Var}X = \frac{a}{a+b}\left[\frac{a+1}{a+b+1} - \frac{a}{a+b}\right]$$

$$= \frac{a}{a+b}\,\frac{(a+1)(a+b) - a(a+b+1)}{(a+b)(a+b+1)}$$

$$= \frac{a}{(a+b)^2(a+b+1)}\cdot\left\{a+b-a\right\} = \frac{ab}{(a+b)^2(a+b+1)}.$$