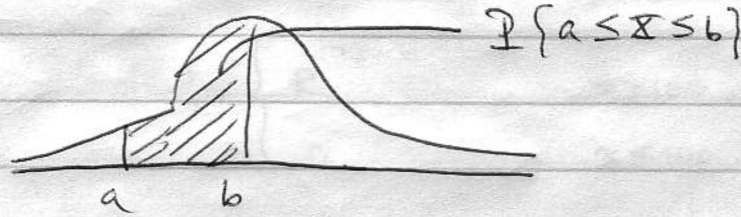
Continuous Random Variables

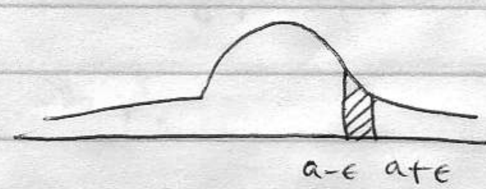
<u>Density function</u> $f \geq 0$ so that

$$P\{X \in A\} = \int_A f(x)\,dx$$



<u>Ex</u> ① $F(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\,dx$, so $F' = f$.

② $P\{X = a\} = \lim_{\epsilon \to 0} \int_{a}^{a+\epsilon} f(x)\,dx = 0$, for <u>all</u> a.



<u>Ex</u> – $f(x) = \begin{cases} C/x^2 & x > 1 \\ 0 & x \leq 1 \end{cases}$ $\quad \int_{-\infty}^{\infty} f(x)\,dx = C \int_1^{\infty} \frac{dx}{x^2} = C$

so $C = 1$.

- Then, e.g., $P\{X \leq 2\} = \int_1^2 \frac{dx}{x^2} = \frac{1}{2}$.

"median"

- $P\{3 < X < 4\} = \int_3^4 \frac{dx}{x^2} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Ex. Life time of a tst tube has pdf

$$f(x) = \begin{cases} \lambda e^{-x} & x > 100 \\ 0 & x \le 100 \end{cases}$$

a) Find λ. $\quad 1 = \lambda \int_{100}^{\infty} e^{-x}\,dx = \lambda e^{-100} \Rightarrow \lambda = e^{100}.$

$$\therefore f(x) = \begin{cases} e^{100-x} & , \quad x > 100, \\ 0 & , \quad x \le 100. \end{cases}$$

b) $P\{X_1 < 120\} = \int_{100}^{120} e^{100-x}\,dx = 1 - e^{-20}.$

Therefore if we pick 3 tst tubes indeptendently, then

$$P\{\text{none lives more than 120 days}\} = P\{X_1 < 120, X_2 < 120, X_3 < 120\}$$

$$= (1 - e^{-20})^3.$$

c) $P\{\text{exactly 1 lives more than 120 days}\}$

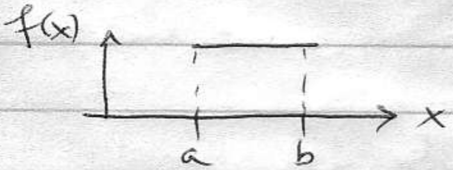
$$= P\{\mathrm{Bin}(3, e^{-20}) = 1\}$$

$$= \binom{3}{1} (1 - e^{-20})^2 (e^{-20}).$$

amples

Uniform Distribution $\quad X$ is uniform (a,b) means

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$



Exponential Distribution $\quad X$ is $\operatorname{Exp}(\lambda)$ (for $\lambda > 0$) means

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Normal Distribution $\quad X$ is Normal(μ, σ^2) or $N(\mu, \sigma^2)$

for $\mu \in \mathbb{R}$ and $\sigma > 0$ means

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty.$$

$f(x) \geq 0$ ✓ , but is $\int_{-\infty}^{\infty} f(x)\,dx = 1$?

Yes, here's why:

$$\left(\int_{-\infty}^{\infty} f(x)\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\,dx\,dy$$

$$= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}-\frac{(y-\mu)^2}{2\sigma^2}}\,dx\,dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{u^2+v^2}{2}}\,du\,dv$$

$$u = \frac{x-\mu}{\sigma}, \quad v = \frac{y-\mu}{\sigma}.$$

Polar coord's:

$$r^2 = x^2+y^2$$
$$dx\,dy = r\,dr\,d\theta$$

$$\therefore \quad \left(\int_{-\infty}^{\infty} f(x)\,dx\right)^2 = \frac{1}{2\pi}\int_0^{2\pi}\Bigg(\underbrace{\int_0^{\infty} e^{-r^2/2}\,r\,dr}_{\int_0^\infty e^{-s}ds = 1}\Bigg)\,d\theta$$

$$= 1. \quad \checkmark$$