Discrete Random Variables

$p(a) = P\{X = a\}$ = mass function $\quad 0 \le p(a) \le 1 \;;\; \sum_a p(a) = 1$

Ex (Binomial trials) $\quad P\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, \dots, n$

Ex – (Neg. Binomial) $\quad$ Bernoulli trials (p)

T = time to the kth success

$$P\{T = \ell\} = \binom{\ell-1}{k-1} p^k (1-p)^{\ell-k} \qquad \ell \ge k$$

Ex – Poisson (λ) $\quad P\{X = k\} = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0, 1, 2, \dots$

$\lambda > 0$ fixed

Check $$\sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{Taylor expansion of } e^{\lambda}} = e^{-\lambda} e^{\lambda} = 1.$$

Random Variables   Formal defⁿ $X : \Omega \longrightarrow \mathbb{R}$ Ω = sample space

[p124]

Example: 3 balls are drawn at random from an urn that contains 3 white, 3 red, 5 black balls.

$1 for each selected ~~red~~ white; else lose $1 for each red.

X = total winnings.   Possible values of X: $0, \pm 1, \pm 2, \pm 3$.

$$P\{X=0\} = \frac{\binom{5}{3} + \binom{3}{1}\binom{3}{1}\binom{5}{1}}{\binom{11}{3}} = 55/165$$

$$P\{X=1\} = \frac{\binom{3}{1}\binom{5}{2} + \binom{3}{2}\binom{3}{1}}{\binom{11}{3}} = P\{X=-1\} = 39/165$$

$$P\{X=2\} = P\{X=-2\} = \frac{\binom{3}{2}\binom{5}{1}}{\binom{11}{3}} = 15/165$$

$$P\{X=3\} = P\{X=-3\} = \binom{3}{3} \Big/ \binom{11}{3} = 1/165$$

check $\sum_{i=-3}^{3} P\{X=i\} = 1.$

$$⊛ = P(X \geq 0) = \frac{55}{165} + \frac{39}{165} + \frac{15}{165} + \frac{1}{165} = \frac{110}{165} = \frac{22}{33} = \frac{2}{3} \simeq 0.66667$$

~~Example ⊛ Redo this if the sampling was done with replacement.~~

~~Now X =~~

~~P(X=0) = ... /11^3~~   ~~Binomial ...~~

~~P(X=0) =~~   ~~$P(X=0) = \frac{5^3 + 3 \times 3 \cdot 3 \cdot 5}{11^3} = \frac{260}{1331}$~~

~~$P(X=1) = P(X=-1) = \frac{3 \times 3 \cdot 5^2 ...}{11^3} = \frac{225}{1331}$~~

~~$P(X=2) = P(X=-2) = \frac{3 ...}{11^3}$~~

3 W   5 B   3 R   3 draws with replacement;

\$1 for each W ; −\$1 for each R.

X = total winnings.

$$P(X=0) = \left(\frac{5}{11}\right)^3 + \left(\frac{3}{11}\right)^2\left(\frac{5}{11}\right) \times 3! = \frac{395}{1331}.$$

$$P(X=1) = P(X=-1) = 3\left(\frac{3}{11}\right)\left(\frac{5}{11}\right)^2 + \left(\frac{3}{11}\right)^2\left(\frac{3}{11}\right) \times \binom{3}{1} = \frac{306}{1331}.$$

$$P(X=2) = P(X=-2) = 3\left(\frac{3}{11}\right)^2\left(\frac{5}{11}\right) = \frac{135}{1331}$$

$$P(X=3) = P(X=-3) = \left(\frac{3}{11}\right)^3 = \frac{27}{1331}.$$

$$P(X \geq 0) = \frac{395}{1331} + \frac{306}{1331} + \frac{135}{1331} + \frac{27}{1331} = \frac{863}{1331} \simeq 0.648385$$

(p125) N distinct types of coupons ; sampling with replacement

T = # of coupons needed to get a complete set of one of each type. Find $P(T=n)$ $\forall n$. Easier to first compute $P(T>n)$.

$A_j = \{$ jth type has not been selected in the 1st n draws $\}$.

Then $P(T>n) = P\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P(A_j) - \sum\sum_{j_1<j_2} P(A_{j_1}A_{j_2}) + \cdots$

$$+ (-1)^{N+1} P(A_{j_1} \cdots A_{j_N}).$$

$$P(A_{j_1}) = \left(1-\frac{1}{N}\right)^n.$$

$$P(A_{j_1}A_{j_2}) = \left(1-\frac{2}{N}\right)^n$$

$$\vdots$$

$$P(A_{j_1} \cdots A_{j_k}) = \left(1-\frac{k}{N}\right)^n.$$

$$\Rightarrow \quad P(T>n) = \sum_{j=1}^{N} \left(1-\frac{1}{N}\right)^n - \sum\sum_{j_1<j_2} \left(1-\frac{2}{N}\right)^n + \cdots$$

$$+ \sum_{j_1<\cdots<j_k} \cdots \sum (-1)^{k+1} \underbrace{P(A_{j_1} \cdots A_{j_k})}_{\left(1-\frac{k}{N}\right)^n}$$

$$+ \cdots + (-1)^{N} \left(1-\frac{N-1}{N}\right)^n + 0.$$

$$= N\left(1-\frac{1}{N}\right)^n - \binom{N}{2}\left(1-\frac{2}{N}\right)^n + \cdots + (-1)^{k+1}\binom{N}{k}\left(1-\frac{k}{N}\right)^n$$

$$+ \cdots + (-1)^N \left(1-\frac{N-1}{n}\right)^n$$

$$= \sum_{j=1}^{N-1} \binom{N}{j}\left(1-\frac{j}{N}\right)^n (-1)^{j+1}.$$

$P(T=n) = P(T>n-1) - P(T>n)$.