Midterm 2 Solutions
Math 5010–1, Spring 2005

1. *An urn contains R red balls and W white balls, where R and W are strictly positive integers. Balls are drawn at random one after another. Every time a ball is drawn, it is replaced back in the urn together with N new balls of the same color.*

   (a) *Compute the probability that a red ball is drawn on the first draw.*

   (b) *Compute the probability that a red ball is drawn on the second draw.*

   Solution: For (a): $P(\mathcal{R}_1) = \boxed{R/(R+W)}$. For (b), use the Bayes rule to find that

$$\begin{aligned} P(\mathcal{R}_2) &= P(\mathcal{R}_2 \mid \mathcal{R}_1)P(\mathcal{R}_1) + P(\mathcal{R}_2 \mid \mathcal{W}_1)P(\mathcal{W}_1) \\ &= \frac{R+N}{R+W+N} \cdot \frac{R}{R+W} \\ &\quad + \frac{R}{R+W+N} \cdot \frac{W}{R+W} = \boxed{\frac{R}{R+W}}. \end{aligned}$$

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2. *In a certain town, 60 percent of all property owners oppose an increase in the property tax while 80 percent of non-property owners favor it. If 65 percent of all registered voters are property owners, then what proposition of registered voters favor the tax increase?*

   Solution: Let $\mathcal{O}$ and $\mathcal{P}$ respectively denote "$\{$oppose increase$\}$" and "$\{$property owner$\}$." Then,

$$\begin{aligned} P(\mathcal{O}) &= P(\mathcal{O} \mid \mathcal{P})P(\mathcal{P}) + P(\mathcal{O} \mid \mathcal{P}^c)P(\mathcal{P}^c) \\ &= (0.6 \times 0.65) + (0.2 \times 0.35) = 0.46. \end{aligned}$$

   Therefore, the answer is $1 - 0.46 = \boxed{0.54}$.

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3. *A pair of fair dice are cast, and the number of rolled dots, on each die, is recorded. Let X denote the maximum of the two numbers. Find the probability mass function of X.*

   Solution: The possible values of X are $1, 2, \ldots, 6$. To find $P(X = 1)$ note that $X = 1$ if and only if we role $(1,1)$; the probabilty is $1/36$. Likewise, $P\{X = 2\}$ is the probability of roling any one of $(2,1)$, $(1,2)$, or $(2,2)$; the chances then are $3/36$. In general, we have $\boxed{P\{X = n\} = (2n-1)/36}$ for $n = 1, \ldots, 6$.

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4. *A fair coin is cast until the first head appears. Let N denote the number of tosses needed to see the first head.*

   (a) *Find the probability mass function of N.*

   (b) *Compute $P\{3 \le N < 8\}$.*

   Solution: Evidently, $\boxed{P\{X = n\} = (1/2)^n}$ for $n = 1, 2, \cdots$. Therefore, $P\{3 \le X < 8\} = P\{X = 3\} + \cdots + P\{X = 7\} = \boxed{0.242}$.

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5. *Here is the simplest mathematical model for the evolution of the price of a commodity: At time zero, the value is zero. Then at every time-step ($n = 1, 2, \cdots$), the stock-price goes up or down by one unit with probability $\frac{1}{2}$. If all stock movements are independent of one another, then compute the probability that the value is zero at time $2n$.*

   Solution: At any time we have the option of going right ('$\rightarrow$') or left ('$\leftarrow$'). The question asks for the probability that, in $2n$ independent trials, we get exactly n $\rightarrow$'s and n $\leftarrow$'s. The answer is $\boxed{\binom{2n}{n}(1/2)^{2n}}$. [This is a question about binomials, why?]
