Math 3210–1, Summer 2016
Solutions to Assignment 4

2.4. #1. (a) $(n+1)^2 - n^2 = 2n+1 > 0$; therefore, $\{n^2\}_{n=1}^{\infty}$ is increasing. It is clearly not bounded above [if it were, then we could find a finite number K such that $n^2 \le K$ for all $n \in \mathbb{N}$; equivalently, $n \le \sqrt{K}$ for all $n \in \mathbb{N}$. But this cannot be, since $\lfloor \sqrt{K} \rfloor + 1$ is a natural number].

(b) $\sqrt{n+1} > \sqrt{n}$ and hence

$$\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \qquad \forall n \in \mathbb{N}.$$

That is, our sequence is decreasing. In particular, $1/\sqrt{n} \le 1$ for all $n \in \mathbb{N}$. Since $1/\sqrt{n} \ge 0$ for all $n \in \mathbb{N}$, our sequence is bounded between 0 and 1 and is therefore a bounded sequence.

(c) Let $a_n = n/2^n$ for all $n \in \mathbb{N}$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)2^n}{n2^{n+1}} = \frac{n+1}{2n} \le \frac{n+n}{2n} = 1 \qquad \forall n \in \mathbb{N}.$$

This proves that $\{a_n\}_{n\in\mathbb{N}}$ is non increasing. Clearly $a_n \ge 0$ for all $n \in \mathbb{N}$. And its monotonicity shows that $a_n \le a_1 = 1/2$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}_{n\in\mathbb{N}}$ is bounded.

(d) Let $a_n = n/(n+1)$ for all $n \in \mathbb{N}$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} > 1 \qquad \forall n \in \mathbb{N}.$$

Therefore, $\{a_n\}_{n\in\mathbb{N}}$ is increasing, whence also satisfies $a_n \ge a_1 = 1/2$ for all $n \in \mathbb{N}$. In the other direction, we have

$$a_n = \frac{n}{n+1} \le \frac{n+1}{n+1} = 1 \qquad \forall n \in \mathbb{N}.$$

Therefore, $\frac{1}{2} \le a_n \le 1$ for all n and hence $\{a_n\}_{n\in\mathbb{N}}$ is bounded.

2.4. #4. $a_n \ge 0$ for all $n \in \mathbb{N}$ because $d_i \ge 0$ for all $i \in \mathbb{N}$. And since $d_i \le 1$ for all $i \in \mathbb{N}$,

$$a_n \le \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n} < 1 \qquad \forall n \in \mathbb{N},$$

using a basic fact about geometric series.

2.4. #5. $s_{n+1} - s_n = a_{n+1} \geq 0$ for all $n \in \mathbb{N}$. Therefore, $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$; that is, $\{s_n\}_{n=1}^\infty$ is non decreasing and hence has a limit thanks to Theorem 2.4.6.

2.5. #1. $I_n = (0\,, 1/n)$ for all $n \in \mathbb{N}$.

2.5. #2. $I_n = [n\,, \infty)$ for all $n \in \mathbb{N}$.

2.5. #5. (a) Any subsequence of $\{a_n\}_{n\in\mathbb{N}}$ contains either infinitely-many terms of the form -2^n $[n \in \mathbb{N}]$ or infinitely-many terms of the form 2^n $[n \in \mathbb{N}]$, or both. In any case we see that no infinite subsequence converges.

(b) For all $n \in \mathbb{N}$,

$$|a_n| \leq \frac{5+n}{2+3n} \leq \frac{5n+3n}{0+3n} = \frac{8}{3}.$$

Therefore, $\{a_n\}_{n=1}^\infty$ is a bounded sequence. We appeal to the Bolzano–Weierstrass theorem [Theorem 2.5.5] to conclude that $\{a_n\}_{n=1}^\infty$ has a convergent subsequence.

(c) $(-1)^n \leq 1$, therefore $0 \leq a_n \leq 2$. This proves that $\{a_n\}_{n=1}^\infty$ is a bounded sequence. We appeal to the Bolzano–Weierstrass theorem [Theorem 2.5.5] to conclude that $\{a_n\}_{n=1}^\infty$ has a convergent subsequence.

2.5. #8. Since $|\sin(n)| \leq 1$ for all $n \in \mathbb{N}$, $\{\sin n\}_{n\in\mathbb{N}}$ is a bounded sequence. By the Bolzano–Weierstrass theorem [Theorem 2.5.5], $\{\sin n\}_{n\in\mathbb{N}}$ has a convergent subsequence.

2.6. #1. (a) $\limsup_{n\to\infty} a_n = 1$, $\liminf_{n\to\infty} a_n = -1$.

(b) $|a_n| = 1/n$ for all $n \in \mathbb{N}$; therefore, $\lim_{n\to\infty} a_n = 0$. By Theorem 2.6.6, $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 0$.

(c) Elementary facts about the sine function tell us that $a_1, a_2, a_3, \ldots$ is just the sequence,

$$\frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2},\ 0,\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ \ldots .$$

Therefore, $\limsup_{n\to\infty} a_n = \sqrt{3}/2$ and $\liminf_{n\to\infty} a_n = -\sqrt{3}/2$.

2.6. #5. For any sequence $\{b_n\}_{n=1}^\infty$,

$$\limsup_{n\to\infty} b_n = \lim_{k\to\infty} \sup_{k\geq n} b_k, \quad \text{and} \quad \liminf_{n\to\infty} b_n = \lim_{k\to\infty} \inf_{k\geq n} b_k.$$

Apply this with $b_n := -a_n$ and use the fact that $\inf_{k\geq n}(-a_k) = -\sup_{k\geq n} a_k$ in order to see that

$$\liminf_{n\to\infty}(-a_n) = \lim_{k\to\infty} \inf_{k\geq n}(-a_k) = \lim_{n\to\infty}\left(-\sup_{k\geq n} a_k\right) = -\lim_{k\to\infty} \sup_{k\geq n} a_k = -\limsup_{k\to\infty} a_k,$$

as was asserted in the problem.