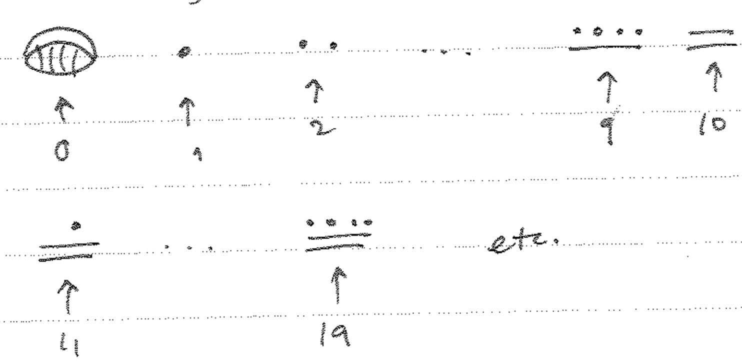
Early Number Systems

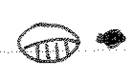
Big / Ancient Idea: combine physical objects & count
→ Tally counts

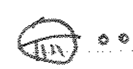
- Tallying (I, II, III, IIII, ~~IIII~~ etc.) $\simeq$ 30,000 B.C. (?)
  - cut notches on bones, sticks, hut walls, etc
  - knots in chords (to mark time; Persian 5 BC –)
    (used as "calculators; Incas ?)

- Abstract Symbols Mayans (300–900 AD ?)
  365 days in a yr.
  18 months of 20 days each
  5 residual days
  ⇒ Number System based on 20.



For 20: $20 = 0\times 1 + 1\times 20$



$40 = 0\times 1 + 2\times 20$  etc.

Egyptian Arithmetic

- Additive ... reduce multiplication to addition.

- Eg. (text) $19 \times 71 = ?$ (Ans $= 1349$)

		$\times 71$
✓	1	71
✓	2	142
	4	284
	8	568
✓	16	1136

↓ double successively
i.e. add to itself

$$19 = 1 + 2 + 16 \Rightarrow 19 \times 71 = (1 \times 71) + (2 \times 71) + (16 \times 71)$$
$$= 71 + 142 + 1136$$
$$= 1349$$

This works in general because every pos. integer k can be written as
$$k_1 \times 1 +$$
$$k_2 \times 2 +$$
$$k_3 \times 4 +$$
$$\vdots$$

Why? (base-2 arith)

- Division = the "converse" of mult. $\frac{91}{7} = ?$ $\Leftrightarrow$ Find x: $7x = 91$

Double 7 until a sum of 91 along the "checks":

1	7	✓
2	14	
4	28	✓
8	56	✓
13	91	

$7 + 28 + 56 = 91$

$1 + 4 + 8 = 13 \Rightarrow 7 \times 13 = 91$

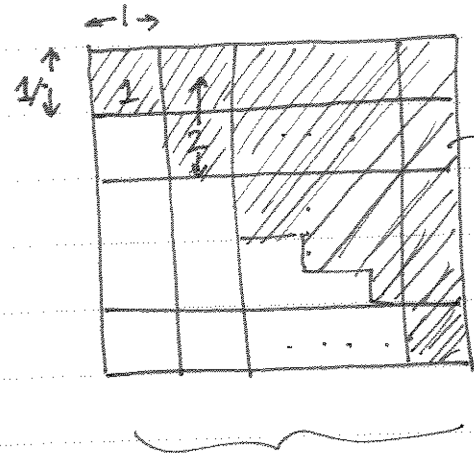
$\frac{91}{7} = 13$ (Why?)

[illegible] also have fractions

Nowadays Formulas
Then Algorithms

Babylonian & Egyptian work on Series

1) Example $1+2+\cdots+n = \frac{(n+1)\,n}{2}$.



$n+1$

area $= 1+2+\cdots+n$

total area $= (n+1)^2$

diagonal $= n+1$

$\therefore 1+\cdots+n = \left[(n+1)^2 - n+1\right]/2$

$= \frac{n+1}{2}\,[\,n\,]$.

OR $1+3+5+\cdots+2n-1 = n^2$ (why?)

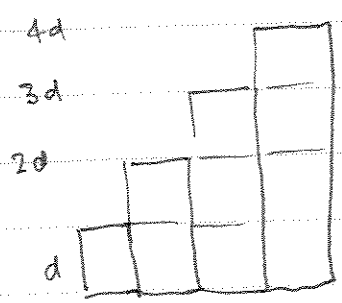
2) Arithmetic Series: Average $a, a+d, a+2d, \ldots, a+nd$

Ans: avg of the endpoints!

$$\text{avg} = \frac{a+(a+nd)}{2} = a + \frac{nd}{2}.$$

$$\therefore \text{sum} = (n+1)\left\{a + \frac{nd}{2}\right\}.$$

$d=1,\ a=0 \longrightarrow (n+1)\frac{n}{2} = 1+\cdots+n$!



or induction.

3) Geometric Series: $a, ar + ar^2 \ldots ar^n$.
(Egyptian)

$S_n = \text{sum} \Rightarrow S_{n+1} = rS_n + a$

but also $S_{n+1} = S_n + ar^{n+1}$

solve ...

Babylonian Mathematics ($\simeq$ 1800–1600 BC)

- Mathematics developed by the people in Mesopotamia (Sumerians, Akkadians, etc.) Today's Iraq region, parts of today's Syria, Turkey, and Iran.

→ Remarkable because their arithmetic was not addition-based.
- Little known; much fairly recently discovered (1930's & on)

→ Sexagesimal # system (base 60) → easier to manipulate fractions
- The original discoverers of, eg., Pythagorean theorem.

→ - They viewed $\frac{x}{y}$ as $x \cdot \frac{1}{y}$, $\frac{1}{y}$ being frequently not hard for them to compute.

<u>Example</u> What is the sexagesimal inverse to 4?

$$\frac{1}{4} = a \cdot \frac{1}{60} + b \cdot \frac{1}{3600} + \ldots$$

<u>Ans:</u> $a = 15 \qquad b = 0$

The Sexagesimal inverse to 5? Answer 12.

—— " —— 7? (Not considered by Babylonians)

$$1/7 = a \cdot \frac{1}{60} + b \cdot \frac{1}{3600} + \ldots$$

Find a by approx.: $1/7 \leq a \cdot \frac{1}{60}$

$$a \geq \cancel{}\, \frac{60}{7} = 8.57142\ldots$$

largest such $a \Rightarrow a = 8$

$$\frac{1}{7} = 8 \cdot \frac{1}{60} + b \cdot \frac{1}{3600} + \ldots \qquad b \geq 34.2857\ldots$$

$\Rightarrow b = 34$ etc.

[illegible] $1/7 = 0;8\ 34\ 17\ 8\ 34\ 17, \ldots$

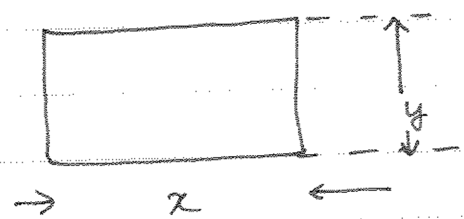
Babylonian Treatment of quadratic Equations

$x^2 = 2$, $x \geq 0$, $x = ?$ [Ans $x = \sqrt{2} \simeq 1.414$]
How?

More generally, $x^2 + ax = b$, $x = ?$

A problem of antiquity: Let R be a rectangle.
How are its area and perimeter length related?

Picture:



$a = x + y$, $b = xy$. Pbm given a and b, find
$\frac{1}{2}$ perimeter, area. x and y.

A modern approach: $x = a - y$, so $b = (a-y)y = -y^2 + ay$.
$y = ?$
This would do the job, since $x = a - y$.

Ex: Find an approx to $\sqrt{2}$ in terms of known objects. (or #s)

(Emulate an old idea)

$x^2 = 2$, $x > 0$.

$1 < x < 2$ because $1 = 1^2 < \underset{2}{x^2} < 4 = 2^2$.

$$\left.\begin{array}{l} 1.4^2 = 1.96 \\ 1.5^2 = 2.25 \end{array}\right\} \Rightarrow 1.4 < x < 1.5$$

etc. ... $\sqrt{2} \simeq 1.41421\cdots$

A generally - Babylonian sol with a modern view

$$x^2 = 2 \longleftrightarrow x^2 - 1 = 1$$

$$\longleftrightarrow (x-1)(x+1) = 1$$

$$\longleftrightarrow x - 1 = \frac{1}{x+1}$$

$$\longleftrightarrow x = 1 + \frac{1}{1+x}$$

$$\Rightarrow \quad x = 1 + \frac{1}{1 + \left(1 + \frac{1}{1+x}\right)} = 1 + \frac{1}{2 + \frac{1}{1+x}}$$

$$= 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

a "continued fraction" representation of $\sqrt{2}$.

As approx:

$$\sqrt{2} = 1 + \frac{1}{2 + \dots} \leq 1 + \frac{1}{2} = 1.5$$

$$\geq 1 + \frac{1}{2+1} = 1.\overline{3}$$

Or $\quad \sqrt{2} = 1 + \dfrac{1}{2 + \frac{1}{2 + \dots}} \geqslant 1 + \dfrac{1}{2 + \frac{1}{2}} = 1.4$!!

Or $\quad \sqrt{2} = 1 + \dfrac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} \leq 1 + \dfrac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$

$$= 1.41\overline{6} \quad !!$$

etc.

Babylonians Solution to Archimedes's formula:

A little calculus: If $x \simeq 0$, then $f(x) \simeq f(0) + f'(0)x$.
Apply this with

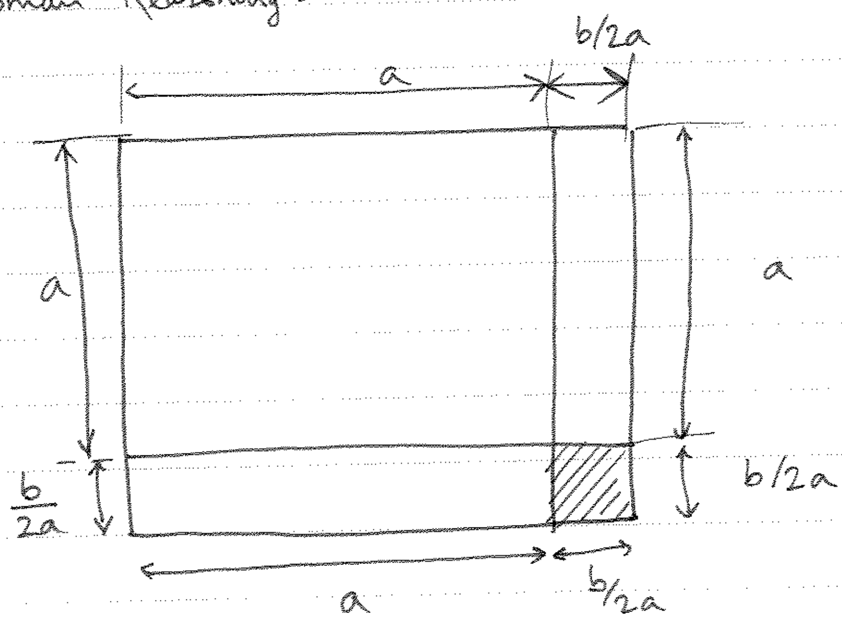
$$f(x) = \sqrt{1+x} \qquad [f(0) = 1]$$
$$f'(x) = \frac{1}{2\sqrt{1+x}} \qquad \left[f'(0) = \frac{1}{2}\right].$$

$$\therefore \quad \sqrt{1+x} \simeq 1 + \frac{x}{2} \quad \text{for } x \simeq 0.$$

Now if $0 < b << a^2$, then

$$\sqrt{a^2+b} = a\sqrt{1 + b/a^2}$$
$$\simeq a\left[1 + \frac{b}{2a^2}\right] = a + \frac{b}{2a}.$$

Babylonian Reasoning:



$$\text{Total area} = \left(a + \frac{b}{2a}\right)^2 = a^2 + \frac{b}{2a}\cdot 2a + \text{area}(\boxed{\ })$$
$$\simeq a^2 + b \qquad \text{if } b \ll a$$

Moreover, this proves that

$$a^2+b \leq \left(a+\frac{b}{2a}\right)^2$$

$$\longleftrightarrow \qquad \sqrt{a^2+b} \leq a+\frac{b}{2a}, \qquad \text{all } a,b>0.$$

Of course also,

$$a \leq \sqrt{a^2+b}. \quad \text{Therefore,}$$

<u>Fact</u> $\quad a \leq \sqrt{a^2+b} \leq a+\frac{b}{2a} \quad$ for all $b,a>0$.

<u>Attempt 1</u>: $\quad a=1,\ b=1 \Rightarrow \qquad 1 \leq \sqrt{2} \leq 1+\frac{1}{2} = 1.5$

<u>Attempt 2</u>: $\quad$ (Need $b \simeq 0$ for better approx.)

$$a=\frac{4}{3} \quad (\cong 1.3) \qquad\qquad b = 2/9$$

$$a^2 = \frac{16}{9} \qquad\qquad a^2+b = \frac{18}{9} = 2$$

$$1.\bar{3} = \frac{4}{3} \leq \sqrt{2} \leq \frac{4}{3}+\frac{1}{12} = \frac{17}{12} = 1.41\bar{6}$$

<u>Attempt 3</u>: $\quad a = 7/5 \ (=1.4) \quad \Rightarrow \quad a^2 = 49/25$

$$b = 1/25$$

$$1.4 = \frac{7}{5} \leq \sqrt{2} \leq \frac{7}{5} + \frac{1/25}{14/5} = \frac{7}{5}+\frac{1}{60} = \frac{17}{12} = 1.41$$

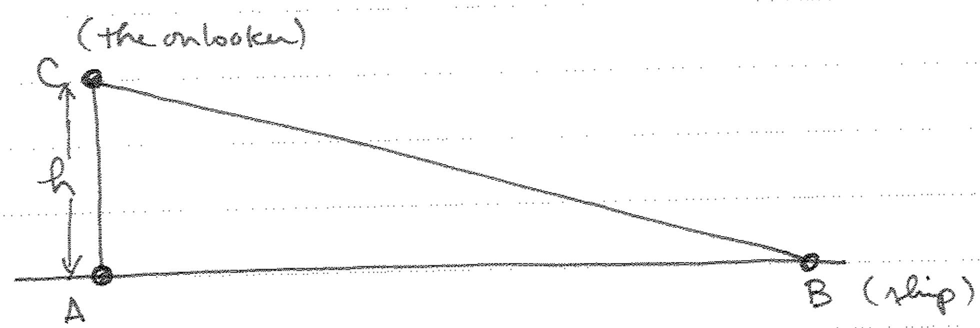
<u>Attempt 4</u>: $\quad a = \frac{141}{100} = 1.41 \qquad\qquad a^2 = \frac{19881}{10{,}000}$

$$1.41 \leq \sqrt{2} \leq \frac{141}{100} + 119/28200 \simeq 1.4142$$

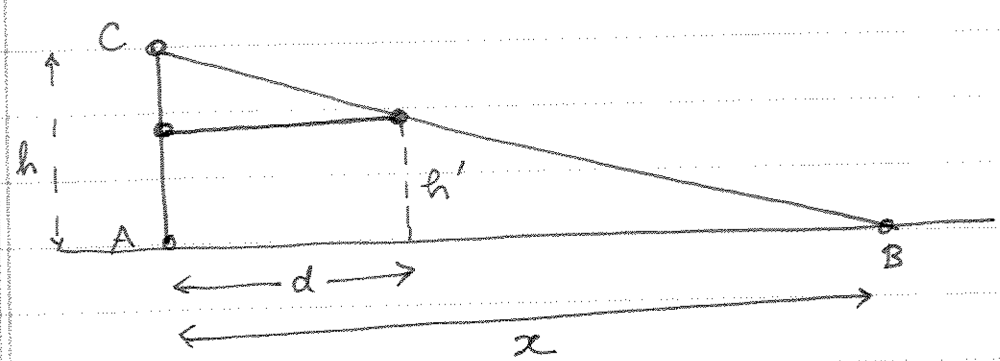
Greek Mathematics ($\simeq$ 6 B.C – 4 B.C)

Thales, Pythagoras, Euclid, Archimedes, Apollonius, Diaphontanus, ...

Thales How far is a ship at sea from the shore? (Thales)



Solution



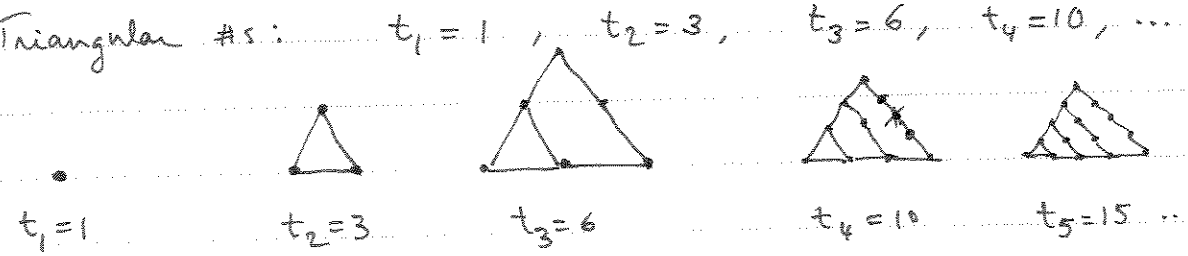
$$\frac{h}{h-h'} = \frac{x}{d}$$

$$\therefore \quad x = \frac{dh}{h-h'} . \checkmark$$

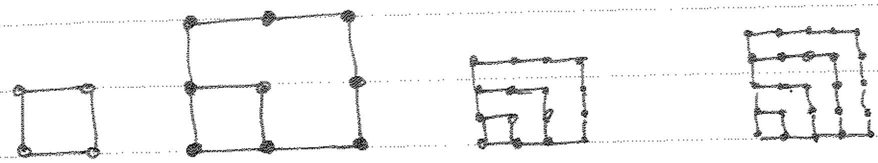
School of Pythagoras

Ex in Number Theory:

Triangular #s: $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, ...



Square #'s $\quad s_1=4,\ s_2=9,\ s_3=16,\ \dots$

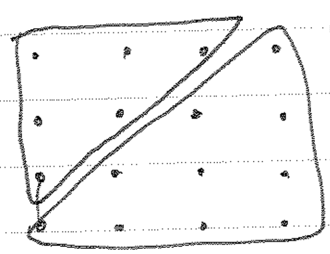


$s_1=4 \qquad s_2=9 \qquad s_3=16 \qquad s_4=25, \dots$

Note $t_n = t_{n-1}+n$. Therefore,

$$\begin{aligned} t_n &= t_{n-1}+n \\ &= (t_{n-2}+n-1)+n \\ &= t_{n-2}+n-1+n \\ &= \dots = 1+2+\dots+n \ ! \end{aligned}$$

What about s_n? $\quad s_n=(n+1)^2$ (clearly)



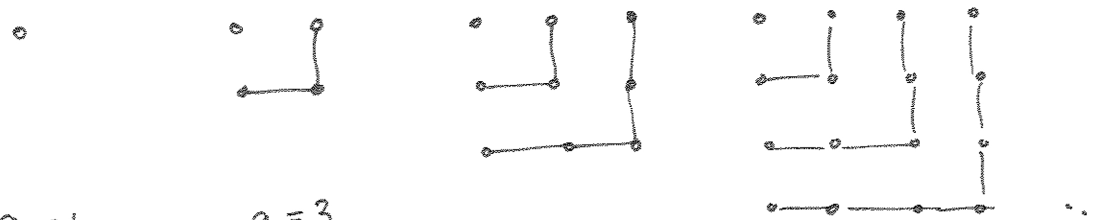
$$\begin{aligned} (n+1)^2 = s_n &= t_n + t_{n+1} \\ &= (1+\dots+n)+(1+\dots+n+1) \\ &= 2(1+\dots+n)+n+1 \end{aligned}$$

$$\therefore \quad 1+\dots+n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

(Yet another method: $1+\dots+n = \frac{1}{2}\times\begin{pmatrix} 1+\dots+n+ \\ n+\dots+1 \end{pmatrix}$
$= \frac{1}{2}n(n+1)$.)

Fruitful idea; e.g.,

L-shaped objects ("gnomons")

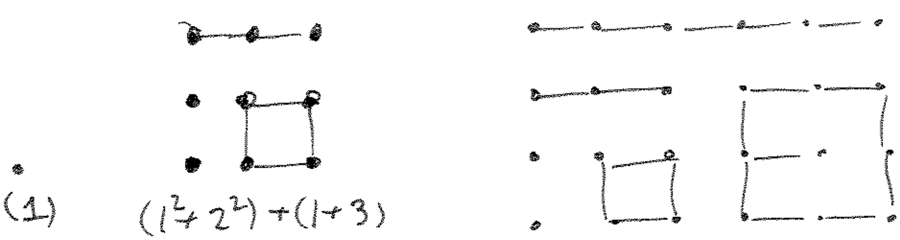


$g_1 = 1$ $g_2 = 3$ $g_3 = 5$ $g_4 = 7$ . . .

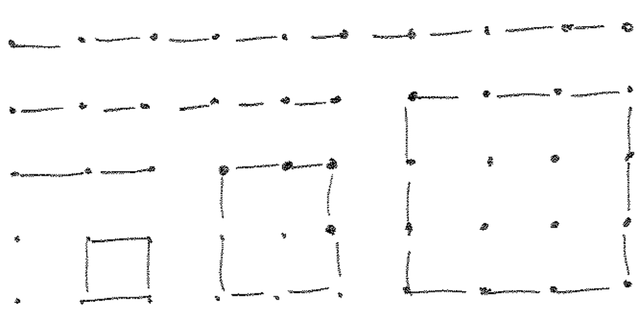
... odd #'s!

$$\Rightarrow \quad 1 + 3 + 5 + \cdots + (2n-1) = n^2 \; !!$$

More sophisticated patterns:



(1) $(1^2 + 2^2) + (1+3)$ $(1 + 2^2 + 3^2) + 1 + 3 + 6$



etc.

$1 + 2^2 + 3^2 + 4^2 + 1 + 3 + 6 + 10$

Method 1

$$1^2 + \cdots + 4^2 + 1 + 3 + 6 + 10 = (1+2+3+4)(4+1)$$

?

$$1^2 + \cdots + n^2 + t_1 + t_2 + t_3 + \cdots + t_n = (1 + \cdots + n)(n+1).$$

$$1 + \cdots + n = \frac{n(n+1)}{2} \Rightarrow$$

$$1^2 + 2^2 + \cdots + n^2 + (t_1 + \cdots + t_n) = \frac{n(n+1)^2}{2}.$$

What is $t_1 + t_2 + \cdots + t_n$?

$$t_1 + \cdots + t_n = \frac{1\times 2}{2} + \frac{2\times 3}{2} + \cdots + \frac{n(n+1)}{2}$$

$$= \frac{1}{2}\left[1\times 2 + 2\times 3 + \cdots + n(n+1)\right]$$

$$= \frac{1}{2}\left[1(1+1) + 2(2+1) + \cdots + n(n+1)\right]$$

$$= \frac{1}{2}\left[1^2 + 2^2 + \cdots + n^2 + 1 + 2 + \cdots + n\right]$$

$$= \frac{1^2 + \cdots + n^2}{2} + \frac{n(n+1)}{4}$$

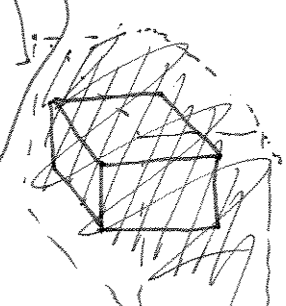
$$\Rightarrow \quad \frac{3}{2}\left(1^2 + \cdots + n^2\right) = \frac{n(n+1)^2}{2} - \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left[2n + 2 - 1\right]$$

$$= \frac{n(n+1)(2n+1)}{4}$$

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad !$$

~~method 2~~



what about cubes?

$$1^3 = 1 = t_1^2$$
$$1^3 + 2^3 = 9 = t_2^2$$
$$1^3 + 2^3 + 3^3 = 36 = t_3^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = t_4^2$$
$$\vdots$$

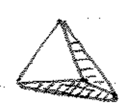
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = t_n^2 = \left[\frac{n(n+1)}{2}\right]^2 \qquad (\simeq 1 \text{ A.D.})$$

Pythagoras to Euclid

Pythagoras (572 - 497 B.C.) :

- 1^{st} known school of Philosophy
- Believed all knowledge is mathematics
- Set the style for Greek philosophy & math

For example, for him the existence of 5 planets (Mercury, Venus, Mars, Jupiter, Saturn) and 5 regular solids wasn't just a coincidence but demonstrated the natural structure of the universe.

1) 

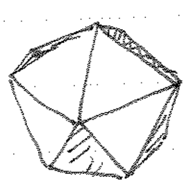
3Δ @ every vertex
(regular) tetrahedron
4 faces
4 vertices

2) octahedron

4 Δ @ every vertex
(8 faces, 6 vertices)

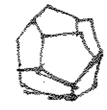
3)

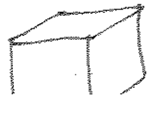
5 Δ @ every vertex



icosahedron (20 faces, 12 vertices)

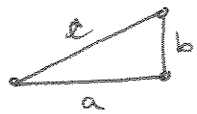
5) Dodecahedron
3 pentagons @ every vertex
(12 faces, 20 vertices)



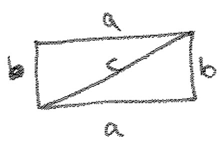
4) 

3 □ @ every vertex
hexahedron ("cube"; 6 faces, 8 vertices)

- Geometric magnitudes were not represented by numbers, and came in distinct types: [Not so for the Asians]
  - line
  - area
  - volume



Modern:
Area $= \frac{1}{2}ab$

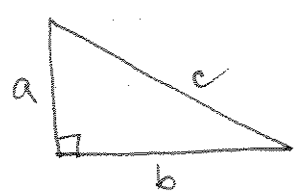
Proof: 

Greek:
Area of  $= \frac{1}{2}$ Area of 

- 2 magnitudes are "commensurable" if they are both integral multiples of a common third magnitude (called a "unit.")
- prior to Pythagoras, people believed there, for any problem, there would be a unit lurking about such that all magnitudes which arose in ~~that~~ problem are integer multiples of this unit.

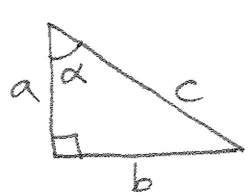
- Pythagoras theorem says this is not so for the hypotenuse & side of an isosceles $\triangle$!!

Theorem (Pythagoras thm)



$$a^2+b^2=c^2.$$

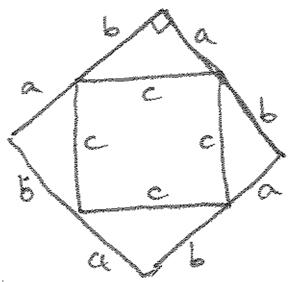
Proof (Modern)



$\cos\alpha = a/c$
$\sin\alpha = b/c$

$$\frac{a^2+b^2}{c^2} = \sin^2\alpha + \cos^2\alpha = 1 \ . \ \#$$

Pf 2 (likely variant of the original proof of Pythagoras)



$$(a+b)^2 = c^2 + 4\cdot\tfrac{1}{2}ab$$

$$a^2+b^2+\underline{\underline{2ab}} = c^2 + \underline{\underline{2ab}}\,. \quad \#$$

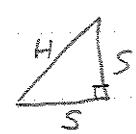
Corollary: the hypotenuse of a right isoceles triangle is incommensurable with its side.

Proof (Aristotle) Suppose H (the hypotenuse) and S (the side) are commensurable. $\Rightarrow$ $\exists$ some unit magnitude U s.t.

$H = \alpha U$

$S = \beta U$ for integers α, β.

We can assume wlog that U is the largest such unit (why?)



$$H^2 = S^2 + S^2$$
$$= 2S^2$$

$\Rightarrow \quad \alpha^2 = 2\beta^2 \quad \Rightarrow \quad \alpha^2$ is even.

If α where odd then α^2 would be too $\Rightarrow$ α is even.

$\Rightarrow \quad \alpha = 2\gamma \quad \Rightarrow \quad 4\gamma^2 = \alpha^2 = 2\beta^2$

$\Rightarrow \quad 2\gamma^2 = \beta^2 \quad \Rightarrow \quad \beta^2$ even $\Rightarrow$ β even.

$\Rightarrow \quad \beta = 2\delta$

$$\Rightarrow \quad \left.\begin{array}{l} H = \underbrace{2\gamma}_{\alpha}\cdot U \\ S = \underbrace{2\delta}_{\beta}\, U \end{array}\right\} \Rightarrow U \text{ isn't maximal!}$$

Contradiction!

An important Aside: (In Euclid's "Elements" -- attributed to Hippasus -- around 5 B.C. -- student of Pythagoras)

Thm $\sqrt{2}$ is irrational.

Pf. Suppose not. $\Rightarrow$ $\exists$ integers a, b s.t. $\sqrt{2} = a/b$.

($\sqrt{2} > 1 \Rightarrow a > b$!!)

Claim If $\sqrt{2} = a/b$ then a, b are even.

Proof $a^2 = 2b^2 \Rightarrow a^2$ even $\Rightarrow$ a even

$\Rightarrow b^2 = \frac{a^2}{2} = \frac{a}{2} \cdot a \Rightarrow b^2$ even $\Rightarrow$ b even. #

$\therefore \sqrt{2} = \frac{a/2}{b/2}$ and $\frac{a}{2}, \frac{b}{2}$ are also even.

$\therefore \sqrt{2} = \frac{a/4}{b/4}$ etc. until $\sqrt{2} = \frac{a/2^N}{b/2^N}$

for even integers $a/2^N$, $b/2^N$ > 1 such that

at lst one of $\frac{a}{2^{N+1}}$, $\frac{b}{2^{N+1}}$ is ≤ 1.

$a > b \Rightarrow \frac{b}{2^{N+1}} = 1 \Rightarrow b = 2^{N+1}$ and so

$\sqrt{2} = a/2^{N+1}$ is an ~~even~~ integer. But

$= \frac{1}{2}\underbrace{\left(\frac{a}{2^N}\right)}_{\text{even}}$ $\qquad 1 < \sqrt{2} < 2$! $\quad \not\#$

Not the most st'd proof. But close to ideas of Aristotle!