Aristotle's view of math more pragmatic: Material objects are the source of reality; one must study the world in order to discover truths.

Plato's physics quite simple: 4 elements (earth, fire, water, air); they are represented by the cube, tetrahedron, icosahedron, and octahedron. The universe is represented by the dodecahedron.

Aristotle's physics starts with Plato's 4 elements, but goes on to explain the cosmos, matter, and motion. Notions of "gravity" (Earth / water) and "levity" (fire / air) are introduced, together with the idea that all things have a "natural motion," determined by gravity and levity they contain. All other motions are "violent." (They are caused by external sources, such as forces of will and soul).

Aristotle's cosmos is finite: A fixed Earth as center, each planet on a concentric (rotating) spherical shell, and a fixed sphere of stars in all. ("firmament")
The planetary spheres rotate at fixed speed; the motive force is natural motion, transferred downward from the firmament.

Aristotle was himself aware that this is an inaccurate picture. (The planets don't move with constant speed; their motion is spatially erratic, even backward at times). The (observed) size of the moon is nonconstant ... Still Aristotle's view was largely accepted.

Aristotle wrestled with the concept of infinity and distinguished b/w "potential infinities" and "actual infinities."
E.g., $\forall$ integers n, $\exists$ a larger integer (say $n+1$). This implies

a potential infinity to Aristotle. But "the set of positive integers" describes an actual infinity.

Aristotle rejected actual infinities, as "they cannot be encompassed by the human mind." But he did embrace potential infinities.

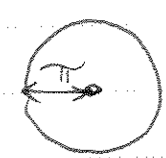
Plato's view of infinitesimals: An atomic theory (time & space are made of indivisible units). Aristotle believed that time & space were infinitely divisible. This led to <u>Zeno's</u> <u>paradox</u>!

<u>The Hellenistic Era</u> ($\simeq$ Pericles to Aristotle)

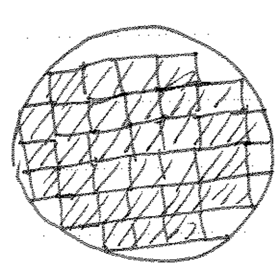
- "Golden Age of Greece"
- Highpoint: Euclid "Elements" and the work of Archimedes (287-212 B.C.)
- We do not know if Euclid is 1, or many, people

Regardless, the viewpoint is remarkably deep, and surprisingly modern, beginning with a set of <u>axioms</u>. Then proceeding to deduce geometry and number theory.

Typical highlight: find π.

<u>Viewpoint</u>: π is the area of 

Estimate it from within and without:



e.g., from within.

Looks like what we do today. But today we "take limits." The Euclidean method was instead:

Pcple of Exhaustion If $A-B<E$ and $B-A<E$ for any magnitude E, then $A=B$.

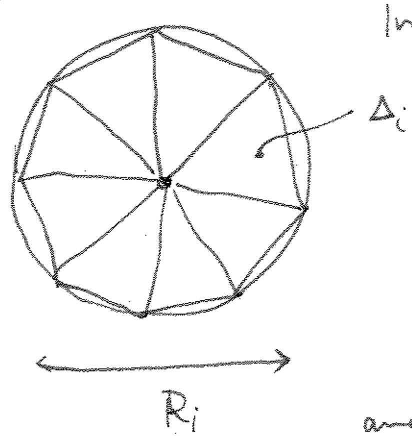
Typical use:

Proposition (Euclid) The areas of circles are to one another as the squares on the diameter.

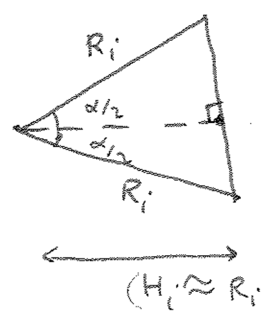
(I.e., if $A(C_i)$ = area of circle of radius R_i then
$A(C_1)/A(C_2) = R_1^2/R_2^2$)

Pf (In modern terms)

Inscribe



$A(\Delta_i) =$



$\alpha \approx 0$

and $A(\Delta_1)/A(\Delta_2) = H_1^2/H_2^2$ (why?)

Apply the pcpl of exhaustion after an "outside" analysis also. #

Proposition (Euclid) Prime #'s are more than any assigned multitude of primes. (∞ - many prime numbers)

Note the statement stating this as a potential infinity.

Pf. Write the 1st n primes for whatever n you have: $p_1, \dots, p_n$. Then $1+\prod_1^n p_i$ is another prime.

Correction 1) Prime # Th $\pi_n \sim \frac{n}{\ln n}$ [not $\frac{1}{\pi}\frac{n}{\ln n}$].

2) Proposition (Euclid) $\exists$ infinitely-many primes.

(Original statement is made in terms of a potential ∞)

Proof Suppose to the contrary that are finitely-many primes, say $p_1 < p_2 < \cdots < p_n$.

Let $x = 1 + p_1 \cdots p_n$.

Clearly, $x > p_n$, so x can't be a prime.

$\Rightarrow$ $\exists$ prime # $p < x$ such that x/p = an integer k.

But p can't be in $\{p_1, \ldots, p_n\}$; for otherwise $p = p_j$ some j $\Rightarrow$

$$\frac{x}{p} = \frac{x}{p_j} = \frac{1}{p_j} + \prod_{\substack{l \neq j \\ 1 \leq l \leq n}} p_l \notin \mathbb{Z}.$$

So the # of primes is $\geq n+1$ and not n. $\times$

They also knew that there are "arbitrarily-large gaps" b/w successive primes:

Consider the string

$n!+2,\ n!+3,\ n!+4, \ldots, n!+n$.

$n!+k$ has k as a factor for every $k \leq n$.

$\Rightarrow$ $\exists$ gap $\geq n-2$ b/w 2 consecutive primes.

Aside (the Eratosthenes "sieve")

goal find $\forall$ primes $\leq n$.

Algorithm: Set $S_0 = \{2, \dots, n\}$

1) Let $p_1 = 2$

2) Remove from $\{2, \dots, n\}$ all integers of the form $p_1 k$ (i.e., $2, 4, 6, 8, 10, \dots$). Set S_1 = the resulting set

3) $p_2 = \min S_1$. [p_2 is a prime]

4) Remove from S_1 all integers of the form $p_2 k$. Set S_2 = the remaining set

etc.

Ex: Find the primes in $\{2, 3, 4, \dots, 20\} \equiv S_0$.

1) $p_1 = 2 \Rightarrow S_1 = \{3, 5, 7, 9, 11, 13, 15, 17, 19\}$

2) $p_2 = 3 \Rightarrow S_2 = \{5, 7, 11, 13, 17, 19\}$

3) $p_3 = 5 \Rightarrow S_3 = \{7, 11, 13, 17, 19\}$

4) $p_4 = 7 \Rightarrow S_4 = \{11, 13, 17, 19\}$

..., $p_5 = 11$, $p_6 = 13$, $p_7 = 17$, $p_8 = 19$. Very fast up to $n \approx 10^8$.

$\simeq$ rise of the Roman empire eminent

<u>Archimedes</u> ($\simeq$ 287 – 212 B.C.)

Conon, Eratosthenes, Appolonius, ...

- Lived in Syracuse (Sicily), but educated by Alexandrians
- Well known for "Eureka, Eureka" [displacement of water]
- Designing war machines; computing trajectories of various projectiles, ...

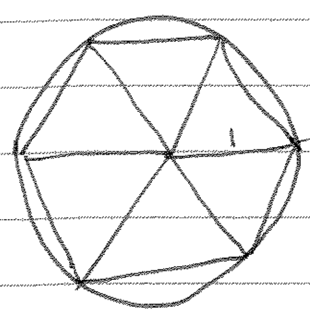
Work in geometry ... ✓

<u>Archimedes Approximation to π</u>

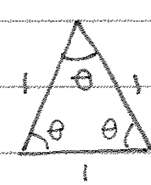
<u>Fact</u>: $\pi \simeq 22/7$ $\left[\begin{array}{l} \pi = 3.1415926\cdots \\ 22/7 = 3.142857\cdots \end{array}\right]$

How? In fact, Archimedes had an algorithm:

± approx



6-sided polygons:



$\theta = 60° \left[= \frac{2\pi}{6}\right]$

area(Δ) = ?

h (Height) =

~~$\Rightarrow$ area of the 6-gon = $\frac{6}{\sqrt{2}}$~~

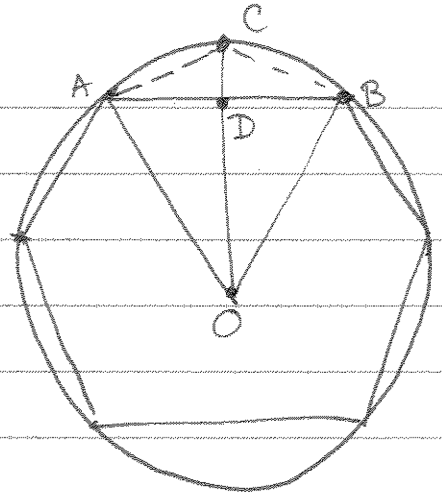
~~$\pi > 6/\sqrt{2} \approx 4.2$~~

$$h^2 + \frac{1}{4} = 1 \Rightarrow h = \sqrt{3/4} = \sqrt{3}/2.$$

$$\text{area}(\Delta) = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{8}$$

$$\text{area}(⬡) = 6 \times \sqrt{3}/8 = 3\sqrt{3}/4 = 1.299013\cdots < \pi$$

2^{nd} approx



Split each angle into 2 to get a 12-gon approx.

$\overline{AB} = 1$ from before.

$\Rightarrow \overline{DB} = 1/2$.

$$\left\{\begin{array}{l} \overline{OD} = \frac{\sqrt{3}}{2} \text{ from before} \\ \overline{OC} = 1 \end{array}\right.$$

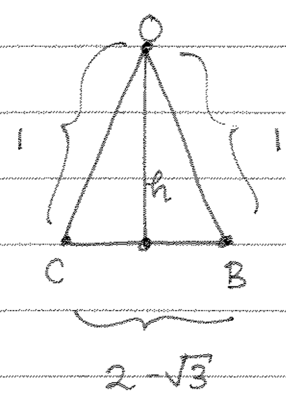
$\Rightarrow \overline{DC} = 1 - \frac{\sqrt{3}}{2}$.

$$\Rightarrow \overline{CB}^2 = \overline{DC}^2 + \overline{DB}^2$$

$$= \left(1 - \frac{\sqrt{3}}{2}\right)^2 + 1/4$$

$$= 1 - \sqrt{3} + 3/4 + 1/4 = 2 - \sqrt{3}$$

$$\Rightarrow \overline{CB} = \sqrt{2-\sqrt{3}}$$



$$h^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2 = 1$$

$$h^2 = 1 - \left(1 - \sqrt{3} + \frac{3}{4}\right)$$

$$= \sqrt{3} - 3/4$$

$\Rightarrow h = \sqrt{\sqrt{3} - 3/4}$ $\Rightarrow$ area$(\overline{OCB}) = \cancel{\frac{1}{2}(2-\sqrt{3})h}$

$$= \frac{1}{2}\sqrt{2-\sqrt{3}}\,\sqrt{\sqrt{3}-3/4}$$

$$\cancel{= h\left(1-\frac{\sqrt{3}}{2}\right)\sqrt{\sqrt{3}-3/4}}$$

area of 12-gon $= 12 \times \frac{1}{2}\sqrt{2-\sqrt{3}}\,\sqrt{\sqrt{3}-3/4}$

$$= 6\sqrt{2-\sqrt{3}}\,\sqrt{\sqrt{3}-3/4}.$$

Next we simplify:

$$\sqrt{2-\sqrt{3}} \cdot \sqrt{\sqrt{3} - 3/4} = \sqrt{(2-\sqrt{3})(\sqrt{3}-3/4)}$$

$$= \sqrt{2\sqrt{3} - 3/2 - 3 + \frac{3\sqrt{3}}{4}}$$

$$= \sqrt{2\sqrt{3} + \frac{3}{4}\sqrt{3} - \frac{9}{2}}$$

$$= \sqrt{\frac{11}{4}\sqrt{3} - \frac{9}{2}} = \frac{1}{2}\sqrt{11\sqrt{3} - 18}$$

$\Rightarrow$ area of 12-gon $= 3\sqrt{11\sqrt{3} - 18}$

$$\simeq 3.077829 < \pi$$

Move to 24, 48, 96, ... -gons for better approximations.

"22/7" comes from a similar analysis of an "outer 96-gon"!

Number Theory

- Diophantus ($\sim$ 200 A.D.) wrote the 1st book on arithmetic
- " introduced symbolic algebra but did little by way of manipulating equations using these symbols [done by Indian & Muslim mathematicians].

- Well known for numerical, not geometric, algorithms for solving specific problems. (By contrast Islamic algebraists expressed rules for manipulating equations, and used those rules in general contexts. But they provided geometric derivations whenever possible)

An Example (In modern notation):

~~Let a, b = given numbers; $\frac{1}{r}$ and $\frac{1}{s}$~~

Consider the following problem:
Divide a given # into 2 #'s such that ~~the~~ given fractions of each number add to another given number.

E.g., a, b = given #'s
$\frac{1}{r}, \frac{1}{s}$ = given fractions

$\Rightarrow$ ~~the~~ Solve for u and v:

$$u + v = a$$
$$\frac{u}{r} + \frac{v}{s} = b$$

Diophantus observes that if the fractions are the same [here, $r=s$] then the second given # must be that fraction of the first. And in that case the "partition" u, v is arbitrary.

Indeed if $r=s$ then $rb=a \Rightarrow u$ and v arbitrary

If $s>r$ then $\frac{a}{s} = \frac{u}{s} + \frac{v}{s}$

$< \frac{u}{r} + \frac{v}{s} = b$

Similarly $b < a/r$. ie,

$$\frac{a}{s} < b < \frac{a}{r}.$$

Concrete ex: $a=100$, $b=30$, $r=3$, $s=5$.

$\Rightarrow \quad u+v=100 \qquad \frac{u}{3} + \frac{v}{5} = 30$

Set $v=5x$: $\begin{cases} u+5x=100 \\ \frac{u}{3}+x=30 \iff u+3x=90 \end{cases}$

$\Rightarrow 2x=10 \rightarrow x=5 \rightarrow v=25 \rightarrow u=75.$

In general, set $v=sx \Rightarrow$

$$\left.\begin{aligned} u+sx&=a \\ u+rx&=rb \end{aligned}\right\} \Rightarrow (s-r)x = a-rb$$

$$\Rightarrow x = \frac{a-rb}{s-r} \Rightarrow \text{solve:}$$

$$u = \frac{s}{s-r}(a - rb) \quad \& \quad v = \frac{r}{s-r}(sb - a).$$

"method of false position" [simplify the p'bm by replacing an unknown by a new one]

Diophantus sol^n to $x^2 = ax + b$. (by the method of false position)

Try $x = y + \alpha \Rightarrow$

$$x^2 = y^2 + 2\alpha y + \alpha^2, \quad ax = ay + a\alpha$$

$$\Rightarrow \quad y^2 + 2\alpha y + \alpha^2 = ay + a\alpha + b.$$

Let $\alpha = a/2 \Rightarrow$

$$y^2 + \alpha^2 = a\alpha + b$$

$$\Rightarrow \quad y = \sqrt{a\alpha + b - \alpha^2}$$

$$= \sqrt{\frac{a^2}{2} + b - \frac{a^2}{4}} = \sqrt{b - a^2/4}.$$

$$\Rightarrow \quad x = \frac{a}{2} + \sqrt{b - \frac{a^2}{4}}$$ "sol^n to quadratic equations"

Aside: In the Middle Ages, this method was used to simplify the cubic equation (!):

$$x^3 = ax^2 + bx + c.$$

Let $x = y + \alpha$:

$$y^3 + 3y^2\alpha + 3\alpha^2 y + \alpha^3 = a(y^2 + 2\alpha y + \alpha^2) + by + b\alpha + c$$

Let $3\alpha = a$:

$$y^3 + \cancel{ay^2} + \frac{1}{3}a^2 y + \frac{a^3}{27}$$

$$= \cancel{ay^2} + \frac{2}{3}a^2 y + \frac{a^3}{9} + by + \frac{ba}{3} + c$$

$$\Rightarrow \quad y^3 = \underbrace{\left(\frac{a^2}{3} + b\right)}_{p} y + \underbrace{\left(\frac{a^3}{9} + \frac{ab}{3} + c - \frac{a^3}{27}\right)}_{q}$$

$$\Rightarrow \quad y^3 = py + q.$$

We'll soon return to such problems!

Europe in the Dark Ages

beginning
≃ 3 century AD : the first barbarian sack of Rome

Toward the end of 3 century A.D. : Emperor Constantine converted to Christianity & moved the capital to Byzanthium, which he renamed "Constantinople" Shortly thereafter, his son, Theodosius, moved the capital back to Rome & adopt Christianity as the official religion of Rome.

Constantine also replaced a system dominated by Roman military power, which was beginning to crumble, by local rule, with oversight from time to time & protection [when necessary] provided by the Roman legions.

The beginning of the feudal system (well established by the time ≃ 411 AD that the Roman Empire fell).

Roman Christianity was basically the same as the pagans' religion. The Christianity of the Middle Ages [and as we know Roman Catholicism today] was built on the thought of the "four fathers" of Christianity in 5 and 6 century A.D. : Ambrose, Augustine, Dominic, & Francis.

St. Augustine (354 - 430 A.D.):

- born in Tunisia ; educated in Alexandria
- lived a "life of earthly pleasures" until he began to sense the futility of such a life
- Embraced Platonism (in the extreme) and incorporated it into Christian theology:
  - (a) Perfection is embodied in everlasting Ideal

which can be approached via contemplation, with scripture as guide.

(b) The only worthwhile effort is the striving for the eternal & the perfect; our existence on earth is transient & base.

(c) The soul is eternal & pure; the body mortal & impure.

- These ideas appeared in "The City of God" — the basic text of the Dark Ages. ~~What~~
- What Europe knew of classical work (that had survived) came via Augustine.
- He explains the trinity
- He Created the doctrine of original sin & redemption through confession
- He Advocated celibacy & austerity

- Augustine's physics was Platonic (or neo-Platonic):
  - The universe is made up of very small indivisible particles of earth, water, air, and fire.
  - Rejected Aristotle's reliance on information gathered via our senses, interpreted strictly by adhering to logic.
  - Rejected Aristotle's concept of continuity & infinite divisibility of space/time.

- St. Augustine wrote that one need only study scripture; other writings are either consonant (hence redundant) with, or contrary (hence heretical) to scripture.
- This set the template for intellectual activity in the Middle Ages.

- During the Dark Ages, there was little use for mathematics: Europe reverted largely to a barter economy. Later on, as national monarchs emerged, more sophisticated math. devices came to use; For instance in England the collection of taxes was recorded by the placement and movement of tokens on a large checkerboard (the work of the Chancellor of the Exchequer.).

The Mathematics of Islam

- At the same time intellectual activity flourished in the East (China, then India, then Islam); esp. after Mohammed (600-649 A.D.). • By the 8th century Islam stretched from Iberia to Indonesia; centers of learning began to appear in all of the major cities.

- The texts of Euclid, Appolonius, Nichomachus, and Ptolemy were translated into Arabic; & formed the basis of math'l learning. Much of our knowledge of those ancient texts (e.g. Ptolemy's) is from translation from the Arabic.

- Al-Khwarizmi (780-850 A.D.) wrote the fundamental work on algebra ("al-jabr w'al muqabala" — "restoring & simplifying").

- Arabic numerals and (today's) algorithms for addition & multiplication became prevalent.
- Al-Khwarizmi wrote on solving linear & quadratic equations, as well as some cubic equations.

- No distinction made b/w numbers and magnitudes. But a computation was always (nearly) accompanied by a geometric interpretation.

- Omar Khayyami's (1048-1131 AD) tie: Maturity and sophistication of algebraic methods.

<u>Example</u> (In today's language): Given $a>b$ find x, y so that $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$.

Today's solution: $\{x^2 = ay \quad , \quad y^2 = xb\}$ ⊛

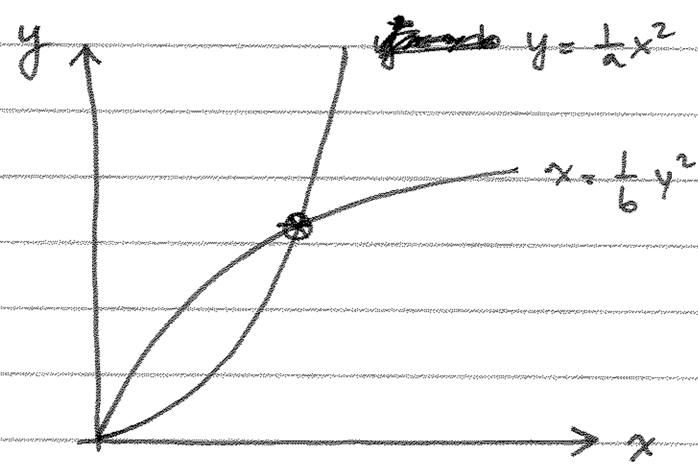
$\Rightarrow \quad y^4 = x^2 b^2 = ab^2 y$

$\Rightarrow \quad y^3 = ab^2$

$\Rightarrow \quad y = (ab^2)^{1/3}$ "hmm".

Omar Khayyami's solution:

⊛ Describes 2 parabolas:



Consider the parabola $y = \alpha x^2$. (α, α) is a point on the parabola; α = the "parameter" of the parabola.

<u>Omar's sol^n</u>: Draw both parabolas on perpendicular axes

with par's a and b. The pt of intersection is the sol^{n}.

- Many of the formulas of algebra appear during this period: The quadratic formula, some cases of the binomial theorem (including Pascal's Δ!), and summation formulas.

<u>Aside on Summation Formulas</u>

E.g., $\displaystyle\sum_{k=0}^{N} r^k = \frac{N(N+1)}{2}$ (geometric series)

$$\sum_{k=0}^{N} (\alpha + \beta k) = (N+1) \cdot \frac{\alpha + (\alpha + N\beta)}{2}.$$

$$\sum_{k=0}^{N} k = \frac{N(N+1)}{2} \qquad \sum_{k=0}^{N} (2k+1) = (N+1)^2$$

$$\vdots$$

[These are in the Elements, w/o the symbols]

$$\circledast \quad \sum_{k=0}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

Here's the Islamic math pf of $\circledast$ ($\simeq$ 9th century AD):

$$(k+1)^3 - k^3 = \left[k^3 + 3k^2 + 3k + 1\right] - k^3$$

$$= 3k^2 + 3k + 1.$$

Add from $k=1$ to $k=N$:

$$\sum_{k=1}^{N} \left[(k+1)^3 - k^3\right] = \sum_{k=1}^{N} (3k^2 + 3k + 1)$$

lhs = "telescoping sum" = $(N+1)^3 - 1$

rhs $= 3\displaystyle\sum_{k=1}^{N} k^2 + 3\sum_{k=1}^{N} k + N = 3\sum_{k=1}^{N} k^2 + \frac{3N(N+1)}{2} + N.$

Solve: $$3\sum_{k=1}^{N} k^2 = (N+1)^3 - 1 - \frac{3N(N+1)}{2} - N$$

$$= (N^3 + 3N^2 + 3N + 1) - 1 - \frac{3}{2}N^2 - \frac{3}{2}N - N$$

$$= N^3 + \frac{3}{2}N^2 + \frac{1}{2}N$$

$$= \frac{N}{2}(2N^2 + 3N + 1) = \underline{\frac{N(N+1)(2N+1)}{2}}.$$

(fact due to Archimedes)

Or (fact due to Al Karaji 11th century AD)

$$\sum_{k=1}^{N} k^3 = \left[\frac{N(N+1)}{2}\right]^2.$$

Note $$(k+1)^4 - k^4 = k^4 + 4k^3 + 6k^2 + 4k + 1 - k^4$$
$$= 4k^3 + 6k^2 + 4k + 1$$

Add from $k=1$ to $k=N$, using telescoping sums:

$$(N+1)^4 - 1 = 4\sum_{k=1}^{N} k^3 + 6\sum_{k=1}^{N} k^2 + 4\sum_{k=1}^{N} k + N$$

$$= 4\sum_{k=1}^{N} k^3 + N(N+1)(2N+1) + 2N(N+1) + N$$

$$N^4 + 4N^3 + 6N^2 + 4N = 4\sum_{k=1}^{N} k^3 + 2N^3 + 3N^2 + N + 2N^2 + 2N + N$$

$$= 4\sum_{k=1}^{N} k^3 + 2N^3 + 5N^2 + 4N$$

Solve. And the method works for computing $\sum_{k=1}^{N} k^j$ for $j = 1, 2, \ldots$ successively.

$j \le 9$

Cavalieri (17th cent. AD)

J. Bernoulli $\forall j$ -- "Bernoulli #'s"

$$A^+ = \sum_{j=1}^{n} \frac{1}{n}\left(j/n\right)^k \quad \geq \text{ area } \geq A^- = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j-1}{n}\right)^k$$

$$\Rightarrow \quad A^+ = \frac{1}{n^{k+1}} \underbrace{\sum_{j=1}^{n} j^k} \qquad A^- = \frac{1}{n^{k+1}} \underbrace{\sum_{j=1}^{n-1} j^k}$$

computes this for $1 \leq k \leq 9$ using Al Karaji's method

Note that $A^+ - A^- = \frac{1}{n^{k+1}} n^k = \frac{1}{n} \longrightarrow 0.$

Ex: $\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \Rightarrow A^+ = \frac{1}{n^2} \frac{n(n+1)}{2} \to \frac{1}{2}$

$k=1$

$\left[\int_0^1 x\,dx = 1/2\right]$

Ex $k=2$ $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \Rightarrow$

$$A^+ = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \to \frac{1}{3}$$

$\left[\int_0^1 x^2 dx = 1/3\right]$

Ex $k=3$ $\sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2 \Rightarrow$

$$A^+ = \frac{1}{n^4} \frac{n^2(n^2+2n+1)}{4} \to \frac{1}{4}$$

$\left[\int_0^1 x^3 dx = 1/4\right]$

etc.

René Descartes (1596–1650)

- All knowledge is attained by pure thought
- No need for experiments / observations
- However, as with Aristotle, Descartes believed that all ~~the~~ agts should adhere strictly to logic.
- Start with a few axioms; prove the rest by logic.
- Rejected Kepler & Galileo's work on astronomy as "occult."
- In math he contributed in several ways;

  Algebra was deemed as central; not geometry!

  (Pure thought)
- Went about to describe, algebraically, Euclid work:
  - A point is a pair of #'s (x, y)
  - A straight line is a relation; $y = Ax + B$
  - A curve is an algebraic relation $y = f(x)$

  The beginnings of the birth of analytic geometry.

Pierre de Fermat (1601–1665)

- Well known for his work on number theory & probability
- Discovered parts of differential calculus
- " how to compute the tangent at a nice curve.
- Didn't relate the 2 topics!

Blaise Pascal (1623–1662)

- Well known for his work on probability (expectation is due to Pascal & Huygens)
- Contributed to the beginnings of calculus:

Pascal & Fermat corresponded over some now famous problems of early probab theory.

Ex. [Pbm of the points] Due to Chevalier de Méré (1654)

- Players A and B play a fair game against one another
  - A needs n games to win ("success")
  - B needs m games to win ("failure")
  - A wins = B loses and vice versa.

What is $P(A \text{ wins})$? ["fair" division of the pot after the game is interrupted after k games]

Pascal's solution Let $P_{n,m}$ = Probab that n ~~games~~ successes before m failures.

$$\Rightarrow \begin{cases} P_{n,m} = \frac{1}{2} P_{n-1,m} + \frac{1}{2} P_{n,m-1} & \forall n,m \geq 1. \\ P_{n,0} = 0 \quad , \quad P_{0,m} = 1 \end{cases}$$

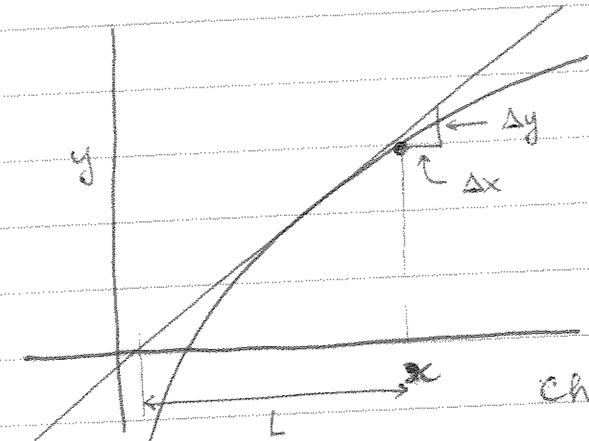
Solve! ("Pascal's Δ")

Fermat's solution (corrected & cleaned up)

In order for n successes to occur before m failures it is NAS that $\exists$ at least n successes in the 1st $m+n-1$ trials.

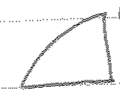
$$\Rightarrow P(A \text{ wins}) = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} \left(\frac{1}{2}\right)^{m+n-1}.$$

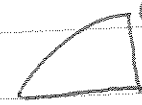
Isaac Barrow (1630-1677)   1st Lucasian prof of Math @ Cambridge



"subtangent"

← "Barrow's infinitesimal triangle"

Change in the area from  (x,y)

to  $(x+\Delta x, y+\Delta y)$

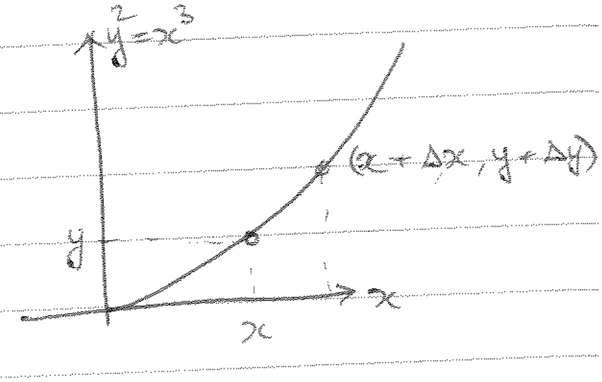
$= ?$

$$\frac{\Delta y}{\Delta x} = \frac{y}{L} \Rightarrow \text{area}(\text{"}\Delta\text{"}) = \frac{1}{2}\Delta x \Delta y = \frac{y}{2L}(\Delta x)^2$$

$\Rightarrow$ Change in area $\simeq y\,\Delta x$.

"fundamental theorem of calculus"

I. Newton was a student in the audience when Barrow was discussing these matters, and suggested an algebraic use of this in order to find the slope.



Ex $y^2 = x^3$

$$(y+\Delta y)^2 = (x+\Delta x)^3$$

$\Downarrow$

$$\underline{\underline{y^2}} + 2y\Delta y + (\Delta y)^2 = \underline{\underline{x^3}} + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$\Rightarrow \quad 2y\,\Delta y + (\Delta y)^2 = 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

Divide by Δx

$$2y\frac{\Delta y}{\Delta x} + \frac{\Delta y}{\Delta x}\cdot\Delta y = 3x^2 + 3x\,\Delta x + (\Delta x)^2$$

$$2y\frac{\Delta y}{\Delta x} \simeq 3x^2 \qquad (\Delta x \simeq 0,\ \Delta y \simeq 0)$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} \simeq \frac{3x^2}{2y} = \frac{3x^2}{2x^{3/2}} = \frac{3}{2}x^{1/2}.$$

i.e., if $y = x^{3/2}$ then $dy/dx = \frac{3}{2}x^{1/2}$!

<u>Isaac Newton</u> <u>(1642-1727)</u>

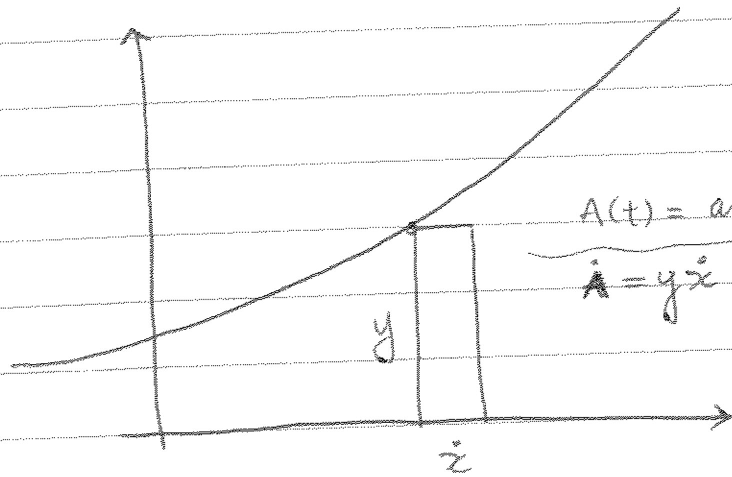
- At the bequest of I. Barrow, Newton wrote up his ideas (1665-6) while Cambridge was closed due to the Great Plague. This work was never published, but is at the heart of Newton's calculus.
- In 1669 Barrow resigned from the Lucasian Chair in favor of Newton, who held it until his death. (1727)

- For Newton calculus is about motion of particles:
  $(x(t), y(t))$ = position of the particle at time $t \geq 0$
  [x and y are "fluents"]

  Velocity $(\dot{x}(t), \dot{y}(t))$ "fluxions"

Goal Compute fluents from fluxions and vice versa.

- A basic tenet [later evolves into Newton's law of motion] is that if the fluxions are constants, then the motion is rectilinear and distance = velocity × time. (falling bodies) In particular, if fluxion = 0 then $\exists$ no motion.



A(t) = area under the trajectory from a fixed point $(x(t), y(t))$

$\dot{A} = y\dot{x}$ [Barrow's Idea]

$\Rightarrow$ Plans of tangent and areas (quadratures) are related: Given $y(t)$ we graph it by taking $x = t$ and consider the area $A(t)$ under the graph. Since $\dot{x} = 1$

$\dot{A} = y$.

$\Rightarrow$ If we had a complete table of all fluxions for all possible fluents then we would read the table in reverse order.

<u>Ex</u> $y = x^n \Rightarrow y + \dot{y} = (x + \dot{x})^n$

$$= x^n + nx^{n-1}\dot{x} + \cdots + (\dot{x})^n$$

$$\Rightarrow \dot{y} = nx^{n-1}\dot{x} + \cdots + (\dot{x})^n$$

$$\dot{y}/\dot{x} = nx^{n-1} + \text{small}$$

- "$\dot{x}$ is nothing in comparison."

$\Rightarrow$ the fluxion of x^n is nx^{n-1} and the fluent of x^n is $\frac{1}{n+1}x^{n+1}$.

Newton believed that all functions can be represented by power series. If so, and if we could apply the preceding rules term by term, then if we had a fluxion $\dot{y}$ as

$$\dot{y} = \sum_{n=0}^{\infty} a_n x^n$$

then the fluent is

$$y = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

The case of the fluxion $1/x$ perplexed him. There is an area under the graph, but he couldn't compute it. Newton's student Taylor solved this problem as follows:

$$\frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n \qquad (0 < x < 1)$$

$\Rightarrow$ the area under $y = \frac{1}{x}$ is $-\sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1}$

$\Rightarrow$ "the antiderivative of $\frac{1}{x}$ is $\ln x$"

Newton & Leibniz discovered rules for differentiation.

For example, suppose the fluents $x, y,$ and z solve $z = xy$. Then

$$z + \dot{z} = (x + \dot{x})(y + \dot{y})$$
$$= xy + \dot{x}y + x\dot{y} + \dot{x}\dot{y}$$

$\Rightarrow$ $\dot{z} = \dot{x}y + \dot{y}x$, since $\dot{x}\dot{y}$ is "as nothing".

Why is $\dot{x}\dot{y}$ "as nothing"? Newton resorts to a trick:

$$z + \tfrac{1}{2}\dot{z} = (x + \tfrac{1}{2}\dot{x})(y + \tfrac{1}{2}\dot{y})$$

$$= xy + \tfrac{1}{2}\dot{x}y + \tfrac{1}{2}x\dot{y} + \tfrac{1}{4}\dot{x}\dot{y}$$

& $$z - \tfrac{1}{2}\dot{z} = xy - \tfrac{1}{2}\dot{x}y - \tfrac{1}{2}x\dot{y} + \tfrac{1}{4}\dot{x}\dot{y}$$

Subtract: $\dot{z} = \dot{x}y + x\dot{y}$.

Bishop Berkeley criticized this claiming that Newton use of such a "trick" showed that Newton himself was not very secure in the foundations of calculus.

- Major work of Newton is "Principia": foundations of dynamics & mechanics; planetary motions.

<u>Gottfried Wilhelm Leibniz</u> (1646 – 1716)

- Leibniz came from money, and was trained in law & philosophy
- His 1st original work were attempts to represent all thoughts symbolically. Enroute he invented "symbolic logic."

- If Newton = a math physicist then Leibniz = a theoretical mathematician
- Instead of fluxions & fluents, Leibniz thought of functional relationships and graphical representations.

- Whereas Newton avoided infinitesimals by relying on an a prior notion of velocity, Leibniz worked directly w/ infinitesimals an in fact created an infinitesimal algebra ("differentials"). Instantantaneous rates were, for Leibniz, ratios of infinitesimals.

- In the early 18th century the Royal Society of London (Newton = President) found Leibniz guilty of plagiarism. Leibniz contentested strongly to the end.

- Today's calculus is due to Leibniz.

E.g. $y = x^n \Rightarrow (y + dy) = (x + dx)^n$

$$= x^n + nx^{n-1}\,dx + \frac{n(n-1)}{2}x^{n-1}(dx)^2 + \cdots$$

In Leibniz's calculus, $(dx)^2 = 0$. So $y + dy = x^n + nx^{n-1}\,dx$

$\Rightarrow dy = nx^{n-1}\,dx$.

E.g., ~~$d(uv) = du$~~ If $w = uv$, then

$$w + dw = (u + du)(v + dv)$$

$$= \underbrace{uv}_{w} + u\,dv + v\,du + \overbrace{du\,dv}^{0}$$

$\Rightarrow dw = u\,dv + v\,du$.

Leibniz & the fundamental theorem of calculus

Let A = a sequence $a, a, \ldots$

Define 2 new sequences:

$$\Sigma(A) = a_0,\ a_0+a_1,\ a_0+a_1+a_2,\ \dots$$
$$\Delta(A) = a_0,\ a_1-a_0,\ a_2-a_1,\ \dots$$

Note $\quad \Sigma(\Delta(A)) = A = \Delta(\Sigma(A))$. ⊛

Now view $y=f(x)$ as a "sequence" indexed by $x \in (c,a)$.
What are the correct analogues of Σ and Δ?

If the [infinitesimal distance] in x's is dx then

$$\cancel{\Sigma(f)} \qquad \Sigma(y) = \int_c^a f(x)\,dx.$$

And ⊛ becomes
$$\int_c^a \frac{dy}{dx}\,dx = a-c$$

$$\frac{d}{da}\int_c^a y\,dx = f(a)\cancel{-y}.$$

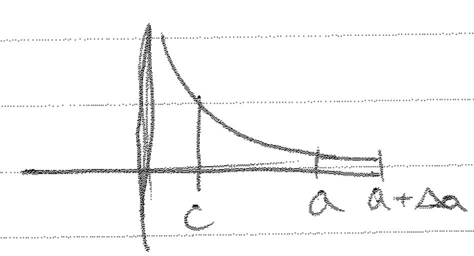
Obtained by discrete approximations.
This discretization has other good uses:

Ex $\quad$ What is $\displaystyle\int_c^a \frac{dx}{x}$?

Modern answer $= \ln a - \ln c$
("log" à la Napier)

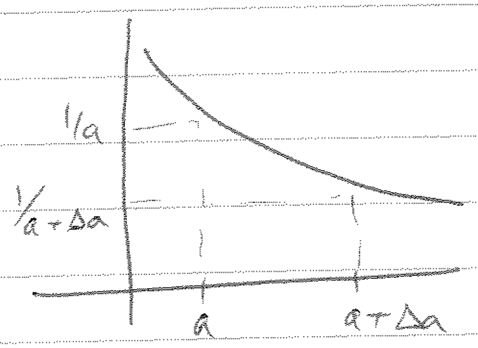
$$A(a) = \int_c^a \frac{dx}{x}.$$

$$A(a+\Delta a) = A(a) + \text{error}$$



$$\text{Error} \simeq \frac{\Delta a}{a+\Delta a} + \frac{1}{2}\Delta a\left(\frac{1}{a} - \frac{1}{a+\Delta a}\right)$$

$$\simeq \frac{\Delta a}{a}.$$



$\Rightarrow$

$$\text{(*)} \quad A(a+\Delta a) - A(a) \simeq \frac{\Delta a}{a}.$$

Now fix a large N. Set $x_0 = c$, $x_1 = c + \frac{c}{N} = x_0\left(1+\frac{1}{N}\right)$

$x_2 = x_1\left(1+\frac{1}{N}\right) \ldots \quad x_{j+1} - x_j = x_j/N \Rightarrow$

$$A(x_{j+1}) - A(x_j) \simeq 1/N.$$

$$A(a) \simeq \sum_{\substack{j \geq 0 \\ x_j \leq a}} \left(A(x_{j+1}) - A(x_j)\right) = \frac{\#(j : x_j \leq a)}{N}$$

$$\text{But } x_k = x_{k-1}\left(1+\frac{1}{N}\right) = x_{k-2}\left(1+\frac{1}{N}\right)^2$$
$$= \ldots = x_{k-l}\left(1+\frac{1}{N}\right)^l$$
$$= x_0\left(1+\frac{1}{N}\right)^k$$
$$= c\left(1+\frac{1}{N}\right)^k$$

$$x_k \leq a \iff c\left(1+\frac{1}{N}\right)^k \leq a$$

$$\left(1+\frac{1}{N}\right)^k \leq a/c$$

$$k \ln\left(1+\frac{1}{N}\right) \leq \ln(a/c) \iff k \leq \frac{\ln(a/c)}{\ln(1+}$$

$\therefore \quad A(a) \simeq \dfrac{\ln(a/c)}{N \ln(1+\frac{1}{N})}$, as $N \to \infty$.

So : ① $N \ln(1+\frac{1}{N}) \xrightarrow{?} \text{const}$; and ② $\displaystyle\int_c^a \frac{dx}{x} = \frac{\ln(a/c)}{\text{const}}$

"natural log" defined s.t. const $= 1$! (Euler)

The birth of probability theory

- Very early 17th century (and all the way back to Antiquity): tied in to "statistical problems," "insurence / actuarial problem" (e.g. mortality rates) and "gambling theory."

- Cardano has rudimentary probablistic analysis of games of chance.
- Pascal and Fermat exchanges ($\simeq$ mid 1600's) started a math. view of probab., as opposed to empirical studies.
- Pascal popularized "math'l induction" (credited in part to Francesco Maurolico of 16th century)

~~Pascal predicted, and in some cases proved ...~~

- James Bernoulli (1654–1705):
    - Nice expository work on Leibniz's work (taught it to his younger brother Johann who taught it to L'Hôpital → Huygens!!)
    - Most famous book is Ars Conjectandi ("The Art of Conjecturing") publish posthumously in 1713 in Latin.
    - Proved the "law of large Numbers" (dubbed later by Poisson):

      Perform [independently & repeatedly] a success/failure trial n ties; each w/ $\mathbb{P}(\text{success}) = p \in (0,1)$.

      let N_n = # of successes. Then $\forall \epsilon > 0$:

$$\mathbb{P}\left(\left|\frac{N_n}{n} - p\right| > \epsilon\right) \longrightarrow 0 \quad \text{as } n \to \infty.$$

Modern proof (due to Pafnutti Lvovich Chebyshev, 1846):

i) $\mathbb{P}(N_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0 \cdots n$ (Fermat-Pascal)

ii) $\sum_{k=0}^{n} k\,\mathbb{P}(N_n = k) = np$ $\quad (= EN_n)$

$\sum_{k=0}^{n} (k-np)^2 \cdot \mathbb{P}(N_n = k) = np(1-p)$ $\quad (= \mathrm{Var}\, N_n)$

iii) $\Rightarrow np(1-p) \geq \sum_{\substack{k=0: \\ |k-np| \geq n\varepsilon}}^{n} (k-np)^2 \mathbb{P}\{N_n = k\}$

$\geq n^2\varepsilon^2 \sum_{-''-} \mathbb{P}(N_n = k) = \varepsilon^2 \mathbb{P}(|N_n - np| \geq n\varepsilon).$

$\Rightarrow$ iv) $\mathbb{P}(|N_n - np| \geq n\varepsilon) \leq \frac{p(1-p)}{n\varepsilon^2} \to 0$ as $n \to \infty$.

Bernoulli argued differently (too complicated), but starts with

$\mathbb{P}(N_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad 0 \leq k \leq n$

$\mathbb{P}(N_{n+1} = k) = \binom{n+1}{k} p^k (1-p)^{n+1-k}$ $\quad 0 \leq k \leq n+1$

$\Rightarrow$ $0 \leq k \leq n \Rightarrow$

$$\frac{\mathbb{P}(N_{n+1} = k)}{\mathbb{P}(N_n = k)} = \frac{\binom{n+1}{k}}{\binom{n}{k}} \cdot (1-p)$$

$$= \frac{(n+1)!\,(n-k)!\,k!}{n!\,(n+1-k)!\,k!} (1-p)$$

$$= \frac{(n+1)}{n+1-k} (1-p)$$

$\left(1 + \frac{k}{\cdot}\right)(1-p)$ "$\downarrow$ as a func of n."

- Abraham de Moivre (1667–1754):
  - French protestant who lived in London after the expulsion of the Huguenots in 1685.
  - Did tutoring & consulting to support himself.
  - Couldn't obtain a professorship because of the divisions caused by the Newton/Leibniz debates.
  - Major work: "Doctrine of Chances" (1718 – subsequent editions into late 1730's)

- Proved Stirling's formula in the following form: ($\simeq 1728$)

As $N \to \infty$,

$$N! \sim C N^{N+\frac{1}{2}} e^{-N} \quad \text{(ratio} \to 1)$$

for $C \approx 2.5066$. Stirling ($\simeq 1730$) proved $C = \sqrt{2\pi}$. In fact, de Moivre proved that

$$N! \approx C N^{N+\frac{1}{2}} e^{-N} \times \left(1 + \frac{1}{12N} - \frac{1}{360N^3} + \frac{1}{1260N^4} \cdots\right)$$

related to "Bernoulli numbers"

A proof of de Moivre's formula

$$f(N) := \frac{N!}{N^{N+\frac{1}{2}} e^{-N}}.$$ Goal: $f(N) \to 1$ as $N \to \infty$.

Approach: Write $f(N)$ as a "telescoping product":

$$f(N) = \frac{f(N)}{f(N-1)} \times \frac{f(N-1)}{f(N-2)} \times \cdots \times \frac{f(2)}{f(1)} \times f(1)$$

$$= e \times \prod_{j=2}^{N} \frac{f(j)}{f(j-1)} = \exp\left[1 + \sum_{j=2}^{N} \left[\ln f(j) - \ln f(j-1)\right]\right]$$

Now evaluate:

$$\ln f(j) = \ln(j!) - \left[(j+\tfrac{1}{2})\ln j - j\right]$$

$$= \sum_{i=1}^{j} \ln i - (j+\tfrac{1}{2})\ln j + j$$

$$\ln f(j-1) = \sum_{i=1}^{j-1} \ln i - (j-\tfrac{1}{2})\ln(j-1) + j - 1$$

$$\Rightarrow \ln f(j) - \ln f(j-1)$$

$$\cancel{= \ln j - j\left[\ln j - \ln(j-1)\right] + \tfrac{1}{2}\left[\ln(j-1) - \ln j\right]} + 1$$

$$\cancel{= \ln j}$$

$$= \ln j - (j+\tfrac{1}{2})\ln j + (j-\tfrac{1}{2})\ln(j-1) + 1$$

$$= -(j-\tfrac{1}{2})\left[\ln j - \ln(j-1)\right] + 1$$

$$= 1 + (j-\tfrac{1}{2})\ln\left(1-\tfrac{1}{j}\right)$$

$$\overset{\text{(Taylor)}}{\approx} 1 + (j-\tfrac{1}{2})\left[-\frac{1}{j} - \frac{1}{2j^2}\right]$$

$$= 1 + \left[-1 - \frac{1}{2j} + \frac{1}{2j} + \frac{1}{4j^2} \right] = \frac{\text{const}}{j^2} \quad \text{.. summable.}$$

So $\quad f(N) \longrightarrow \exp\left[1 + \sum_{j=2}^{\infty} \left[\ln f(j) - \ln f(j-1) \right] \right]$. #

- Stirling's contribution ("Methodus Differentialis" 1730)
  (Method due to P.-S. Laplace 1800)

i) Define $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} \, ds$ for $\alpha \geq 0$
"gamma fnc"

ii) $\Gamma(0) = \infty$, $\Gamma(1) = 1$

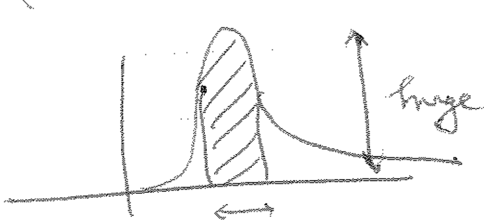
iii) $\Gamma(\alpha) = (\alpha-1)\,\Gamma(\alpha-1)$ for $\alpha > 1$.

(So $\Gamma(2) = 1 \times \Gamma(1) = 1$, $\Gamma(3) = 2 \times \Gamma(2) = 2 \times 1$
.. $\Gamma(k) = (k-1)!$)

iv) Write

$$\Gamma(\alpha) = \int_0^\infty \exp\left(-s + (\alpha-1)\ln s \right) ds.$$

Idea If α large then $\exp(\downarrow)$ is huge on a small set (where the exponent is near its max.)



Write $\Gamma(\alpha) = \int_0^\infty e^{h(s)}\, ds$ for

$h(s) = (\alpha-1)\ln s - s$ and note:

- $h'(s) = \frac{\alpha-1}{s} - 1 \Rightarrow h' = 0 \to s = \alpha - 1$

- $h''(s) = \frac{-(\alpha-1)}{s^2} \le 0$ for α large $\Rightarrow$ max

"$\Rightarrow$" $\Gamma(\alpha) \approx \int_{s \approx \alpha-1} e^{h(s)}\, ds$ $\longleftarrow$ $1-\varepsilon \le \frac{s}{\alpha-1} \le 1+\varepsilon$ small but fixed ε (say $\varepsilon = 1/2$)

If $s \approx \alpha - 1$, then

$$\cancel{h(s) \approx h(\alpha-1) + (\alpha-1)h'(\alpha-1) + \frac{1}{2}(\alpha-1)^2 h''(\alpha-1)}$$

$$\cancel{= (\alpha-1)\ln(\alpha-1) - (\alpha-1) + 0 - \frac{1}{2}(\alpha-1)^2 \cdot \frac{\alpha-1}{(\alpha-1)^2}}$$

$$\cancel{= (\alpha-1)\ln(\alpha-1) - \frac{3}{2}}$$

$$h(s) \approx h(\alpha-1) + (\alpha-1)\overbrace{h'(\alpha-1)}^{0} + \frac{1}{2}(s-\alpha+1)^2 h''(\alpha-1)$$

$$= (\alpha-1)\ln(\alpha-1) - (\alpha-1) - \frac{1}{2}(s-\alpha+1)^2 \cdot \frac{1}{(\alpha-1)}$$

$$\therefore \Gamma(\alpha) \approx \int_{(\alpha-1)(1-\varepsilon)}^{(\alpha-1)(1+\varepsilon)} e^{(\alpha-1)\ln(\alpha-1) - (\alpha-1) - \frac{1}{2}(s-\alpha+1)^2/(\alpha-1)}\, ds$$

$$= e^{-(\alpha-1)} (\alpha-1)^{\alpha-1} \int_{(\alpha-1)(1-\varepsilon)}^{(\alpha-1)(1+\varepsilon)} e^{-\frac{1}{2}[s-(\alpha-1)]^2/(\alpha-1)}\, ds$$

$$\Gamma(\alpha+1) \simeq e^{-\alpha}\alpha^{\alpha} \int_{\alpha(1-\epsilon)}^{\alpha(1+\epsilon)} e^{-\frac{1}{2}[s-\alpha]^2/\alpha}\, ds$$

$$= e^{-\alpha}\alpha^{\cancel{\alpha}} \int_{-\epsilon\sqrt{\alpha}}^{+\epsilon\sqrt{\alpha}} e^{-t^2/2}\, dt\, \sqrt{\alpha} \qquad \left(t = \frac{s-\alpha}{\sqrt{\alpha}}\right)$$

$$= e^{-\alpha}\alpha^{\alpha+\frac{1}{2}} \underbrace{\int_{-\epsilon\sqrt{\alpha}}^{+\epsilon\sqrt{\alpha}} e^{-t^2/2}\, dt}$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$$

<u>Aside on Laplace's Method</u> Estimate $I_n = \int_0^{\pi} (1+\cos x)^n\, dx$ as $n \to \infty$.

write $I_n = \int_0^{\pi} e^{f(x)}\, dx$, where

$$f(x) = n \ln(1+\cos x).$$

$$f'(x) = \frac{-n}{1+\cos x} \sin x$$

$$f''(x) = \frac{-n\cos x(1+\cos x) - n\sin^2 x}{(1+\cos x)^2}$$

$$= \cancel{n\cos x} - \frac{n(\cos x+1)}{(1+\cos x)^2} = -\frac{n}{1+\cos x}$$

($<0 \rightarrow$ unique max at $x=0$)

$f(0) = n\ln 2 \qquad f''(0) = -n/2$

$$\therefore \quad I_n \approx \int_0^{\varepsilon} \exp\Big[f(0) + \overbrace{\cancel{x f'(0)}}^{0} + \frac{x^2}{2} \overbrace{f''(0)}^{-n/2}\Big]\,dx$$

$$= \int_0^{\varepsilon} \exp\left[n \ln 2 - \frac{n x^2}{4}\right] dx$$

$$= 2^n \int_0^{\varepsilon} \exp\left[-\frac{n x^2}{4}\right] dx$$

$$= 2^n \int_0^{\varepsilon n/2} \exp(-y^2/2)\,dy\,\sqrt{\frac{2}{n}} \qquad \left(y = \sqrt{\frac{n x^2}{2}}\right)$$

$$\approx \frac{2^{n+\frac{1}{2}}}{\sqrt{n}} \underbrace{\int_0^{\infty} e^{-y^2/2}\,dy}_{\sqrt{\frac{\pi}{2}}}$$

$$= \sqrt{\frac{\pi}{2n}}\; 2^{n+\frac{1}{2}}.$$ (comes up in "Fourier Analysis")

ack to
e Moivre

– Most famous works of de Moivre: ① CLT (also ascribed to Laplace .. incorrectly)
② Generating fncs

On the CLT: N_n, as in Bernoulli, = # of success in n p-coin trials.

$$\mathbb{P}(N_n = k) \approx \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \le k \le n$$

Heuristically,

$$\mathcal{P}_n \triangleq \mathbb{P}\left(N_n - np \approx \sqrt{np(1-p)}\, x\right)$$

$$\text{“}\approx\text{”} \quad \binom{n}{np + \sqrt{np(1-p)}\,x} \; p^{np+\sqrt{np(1-p)}\,x} \, (1-p)^{n - \text{''}}$$

$q := 1-p$.

$$\mathcal{P}_n = \frac{n!}{(np + \sqrt{npq}\,x)!\,(nq + \sqrt{npq}\,x)!} \; p^{np+\sqrt{npq}\,x} \; q^{nq - \sqrt{npq}\,x}$$

and apply stirling:

$$\mathcal{P}_n \approx \frac{e^{-x^2/2n}}{\sqrt{2\pi n}}$$

$$\text{“}\Rightarrow\text{”} \quad \mathbb{P}\left(a \le \frac{N_n - np}{\sqrt{npq}} \le b\right) \approx \sum_{a \le x \le b} \frac{e^{-x^2/2n}}{\sqrt{2\pi n}}$$

$$\approx \int_a^b \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

An arithmetization of analysis (After Gauss, Cauchy, Lagrange, Dirichlet, etc.)

- J. B. J. (Joseph) Fourier (1768 - 1830)
- Karl Weierstrass (1815-1897) Univ. of Berlin
- Georg Cantor (1845-1918) " " Halle
- J. W. R. (Richard) Dedekind (1831-1916) " " Braunschweig

We close by saying a few things about "the most important" name here: G. Cantor.

Basic Question (of Antiquity, eg. Plato) What is "infinity"?

- Descriptive Set theory: think of "sets" via their "cardinality."

~~Identify sets according to their cardinality.~~
~~$\emptyset$ a set with 0 elements~~

Start with logic; then create "numbers":

$0 \hat{=}$ cardinality of $\emptyset$ (Defining "cardinality")

$1 \hat{=}$ " " $\{\emptyset\} = 1$ $(|\emptyset| = 1)$

$2 \hat{=}$ " " $\{\emptyset, \{\emptyset\}\}$ $(|\emptyset, \{\emptyset\}| = 2)$

$3 \hat{=}$ " " $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ ⋮

etc. (due to Frege, roughly)

$\mathbb{N} = \{1, 2, 3, \ldots\}$ $|\mathbb{N}| := \aleph_0$ (aleph-zero) ← A countable if $|A| \le \aleph_0$

$|\mathbb{R}| := c$ (continuum)

- algebra of cardinals similar to algebra on $\mathbb{R}$; notably

$$A^B := \text{all fncs } f: B \to A \Rightarrow$$

$$|A^B| = |A|^{|B|}. \quad \text{E.g. } \left|\{1,2\}^{\{0,1,3\}}\right| = 2^3 = 8.$$

- Note $A^B = \{f: B \to A\}$ so $A^{\mathbb{N}} \cong$ all infinite sequence $(a_1, a_2, \dots)$, $a_i \in A$.
- $|A| \le |B| \iff \exists$ fnc $f: A \to B$. Therefore,

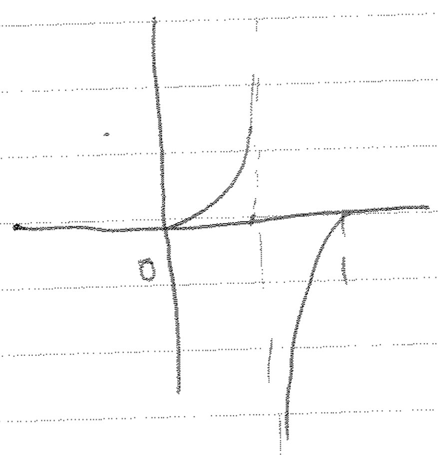
  $|A| = |B| \iff \exists$ 1-1 onto f such that

  $f: A \to B$ & $f^{-1}: B \to A$.

<u>Theorem</u> (Cantor) $c = 2^{\aleph_0}$

$$\left(\iff |\mathbb{R}| = \left|\{0,1\}^{\mathbb{N}}\right|\right)$$

<u>Pf</u> <u>Step 1</u> let $f(x) = \tan(\pi x)$ $\quad 0 < x < 1$.



$f: (0,1) \xrightarrow{\text{1-1 onto}} \mathbb{R}$!

So $|(0,1)| = |\mathbb{R}|$.

<u>Step 2</u> Enough to prove $|(0,1)| = |\{0,1\}^{\mathbb{N}}|$.

Every $0 < x < 1$ has a unique binary expansion and vice versa:

$$x = 0 + \frac{x_1}{2} + \frac{x_2}{4} + \frac{x_3}{8} + \dots \quad \left(\begin{matrix}\text{use the terminating one} \\ \text{for dyadic rationals}\end{matrix}\right)$$

Then define

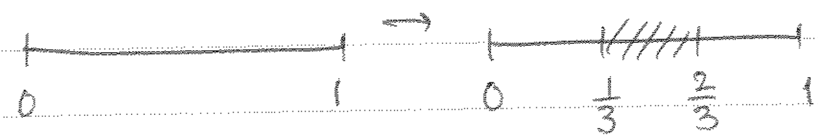
$$f(x) = (x_1, x_2, x_3, \cdots)$$

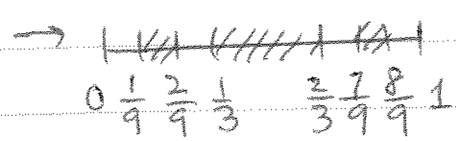
$$f : (0,1) \to \{0,1\}^{\mathbb{N}} \quad \text{1-1 onto.} \quad \#$$

Q. What is an integral? (Goes back to Ancients via Riemann)

↑ fund thm of calculus

Cantor's set:



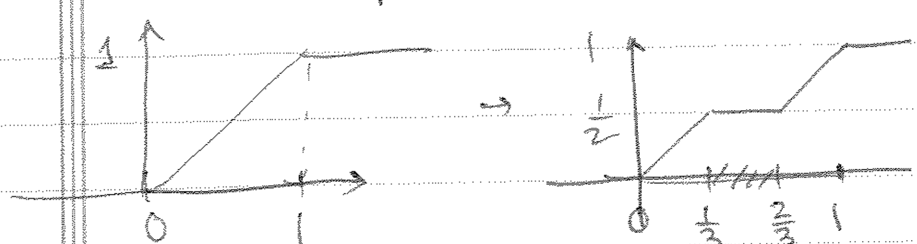


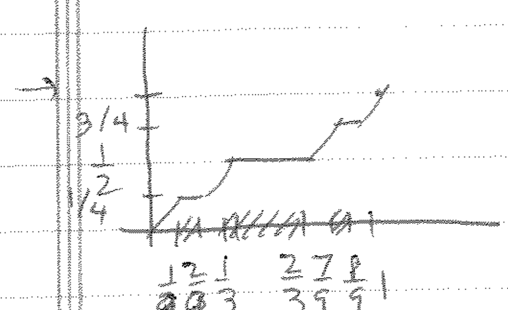
$\text{leb}(C_n) = 3^{-n} \times 2^{+n}$

$\to 0.$

$C = \bigcap_n C_n \to \text{leb}(C) = 0$

Cantor-Lebesgue fnc:





fact $\lim_n f_n(x) = f(x)$ exists

& is cont. $f_n'(x) = 0$

& $f_n(x) = f(x)$ $\forall$ x off of C_n ~~[illegible]~~

so $f'(x) = 0$ off of C — a set of 0 Leb. m

$1 = f(1) - f(0) \neq \int_0^1 f'(x)\,dx \longrightarrow$ leads to modern integration thy