SCHOENBERG'S THEOREM VIA THE LAW OF LARGE NUMBERS (NOT FOR PUBLICATION)

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Abstract. A classical theorem of S. Bochner states that a function $f : \mathbf{R}^n \to \mathbf{C}$ is the Fourier transform of a finite Borel measure if and only if f is positive definite. In 1938, I. Schoenberg found a beautiful complement to Bochner's theorem. We present a non-technical derivation of of Schoenberg's theorem that relies chiefly on the de Finetti theorem and the law of large numbers of classical probability theory.

1. Introduction

A real-valued function g of n vectors is said to be *positive semi-definite* (sometimes, *positive definite*) if $\sum_{i=1}^k \sum_{j=1}^k g(x_i - x_j)c_i\bar{c}_j \geq 0$ for all n-vectors $x_1, \ldots, x_k$ and all complex numbers $c_1, \ldots, c_k$.

A classical theorem of S. Bochner (1955, Theorem 3.2.3, p. 58) asserts that positive semi-definite functions are precisely those that are Fourier transforms of finite measures. Let $\|\cdot\|_n$ denote the usual Euclidean norm in n dimensions. That is, $\|x\|_n = (x_1^2 + \cdots + x_n^2)^{1/2}$ for all $x \in \mathbf{R}^n$. Then, the goal of this note is to present a very simple proof of the following well-known theorem of I. J. Schoenberg (1938, Theorem 2):

Schoenberg's Theorem. *Suppose $f : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous. Then, the following are equivalent:*

1. *The function $\mathbf{R}^n \ni x \mapsto f(\|x\|_n)$ is positive semi-definite.*
2. *The function $\mathbf{R}_+ \ni t \mapsto f(\sqrt{t})$ is the Laplace transform of a finite Borel measure on $\mathbf{R}_+$.*

Originally, this theorem was used to describe isometric embeddings of Hilbert spaces. Since its discovery, it has also found non-trivial connections to other diverse areas ranging from classical, as well as abstract, harmonic analysis (Berg and Ressel, 1978; Berg et al., 1984; Kahane, 1985) to the measure theory of Banach spaces

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(Bretagnolle et al., 1965; 1965/1966; Bretagnolle et al., 1967; Christensen and Ressel, 1982; Koldobsky, 1996; Misiewicz, 1996a; 1996b; Koldobsky, 1999; Koldobsky and Lonke, 1999), function theory (Ressel, 1974; Diaconis and Freedman, 2004a; 2004b) and to the foundations of statistics via de Finetti-type theorems (Freedman, 1963; Ressel, 1985; Diaconis and Freedman, 2004a; 2004b). For other relations, in particular, to statistical mechanics, see the detailed historical section of Diaconis and Freedman (2004b).

Although Schoenberg's original proof is not too difficult to follow, it is somewhat technical. P. Ressel (1976) has devised a simpler proof which rests on a characterization of Laplace transforms (Ressel, 1974, Satz 1) that is similar to Schoenberg's theorem. We are aware also of another simple proof, due to J. Bretagnolle, D. Dacuhna–Castelle, and J.-L. Krivine (1965; 1965/1966; 1967). Their proof is similar to the one presented here, but is slightly more technical.

The present article aims to describe a self-contained, elementary, and brief derivation of Schoenberg's theorem. Our proof assumes only a brief acquaintance with real analysis and measure-theoretic probability theory. This proof is quite robust and can be used to produce more general results; all one needs is a more general setting in which a basic form of the de Finetti theorem and the law of large numbers hold.

Since writing the first draft of this paper, we have found out about the work of D. Kelker (1970, Theorem 10). Kelker's proof is essentially the same as ours. J. Kingman (1972) contains yet another rediscovery of Kelker's proof.

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2. The Proof

All notation and references to probability theory are standard and can be found in any standard first-year graduate textbook.

Without loss of generality, we may suppose that $f(0) = 1$. Then, thanks to Bochner's theorem, Schoenberg's theorem translates to the equivalence of the following two assertions:

(1°) For all $n \geq 1$ there exists a Borel probability measure μ_n on $\mathbf{R}^n$ such that

$$f\left(\sqrt{\sum_{i=1}^n x_i^2}\right) = \int_{\mathbf{R}^n} e^{ix\cdot y}\,\mu_n(dy) \quad {}^\forall x := (x_1, \ldots, x_n) \in \mathbf{R}^n. \tag{2.1}$$

(2°) There exists a Borel probability measure ν on $\mathbf{R}_+$ such that

$$f(t) = \int_0^\infty e^{-t^2 s/2}\, \nu(ds) \qquad {}^\forall t > 0. \tag{2.2}$$

Therefore, it suffices to prove that (1°) and (2°) are equivalent. The assertion, "(2°)⇒(1°)" follows from a direct computation because $\|x\|_n \mapsto \exp(-\|x\|_n^2 s/2)$ is manifestly a Fourier transform on $\mathbf{R}^n$. So we prove only the converse. Henceforth, we assume that (1°) holds.

Our next lemma follows immediately from (1°) and the uniqueness theorem.

Lemma 1. *The family $\{\mu_n\}_{n=1}^\infty$ is consistent.*

It might help to recall that "$\{\mu_n\}_{n=1}^\infty$ is *consistent*" means that for all $n \geq 1$ and all linear Borel sets $A_1, A_2, \ldots$, $\mu_n(A_1 \times \cdots \times A_n) = \mu_{n+1}(A_1 \times \cdots A_n \times \mathbf{R})$.

Proof of Schoenberg's Theorem. In accord with Lemma 1 and the Kolmogorov consistency theorem, there exists an exchangeable stochastic process $\{Y_k\}_{k=1}^\infty$, on some probability space $(\Omega, \mathscr{F}, \mathrm{P})$, such that for all $n \geq 1$ and all Borel sets $A \subset \mathbf{R}^n$,

$$\mathrm{P}\{(Y_1, \ldots, Y_n) \in A\} = \mu_n(A). \tag{2.3}$$

Choose and fix some $t > 0$, and introduce a sequence $\{X_i\}_{i=1}^\infty$ of independent random variables such that every X_i has the normal distribution with mean 0 and variance t^2. We can assume, without loss of generality, that the X_i's are defined on the same probability space $(\Omega, \mathscr{F}, \mathrm{P})$. We first apply (1°) with $x := n^{-1/2}(X_1, \ldots, X_n)$, and then take expectations, to deduce that for all $n \geq 1$,

$$\begin{aligned} \mathrm{E}\left[f\left(\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2} \right) \right] &= \int_{\mathbf{R}^n} \exp\left(-\frac{t^2\|y\|_n^2}{2n} \right) \mu_n(dy) \\ &= \mathrm{E}\left[\exp\left(-\frac{t^2}{2n}\sum_{i=1}^n Y_i^2 \right) \right]. \end{aligned} \tag{2.4}$$

See (2.3) for the last identity. Now let $n \to \infty$. The simplest form of the law of large numbers dictates that $\sum_{i=1}^n X_i^2/n \to \mathrm{Var}X_1 = t^2$ in probability. Therefore, the left-hand side of (2.4) converges to $f(t)$ by the dominated convergence theorem.

By the de Finetti theorem, the Y_i's are conditionally i.i.d. given the exchangeable σ-algebra generated by the Y_i's. Thanks to the Kolmogorov strong law of large numbers, and by the Fubini–theorem, $L := \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n Y_i^2$ exists a.s. Moreover, the event $\{L < \infty\}$ agrees upto null sets with $\{\mathrm{E}[Y_1^2 \,|\, \mathscr{E}] < \infty\}$, where $\mathscr{E}$ denotes the exchangeable σ-algebra of $\{Y_i\}_{i=1}^\infty$. By the dominated convergence theorem, the right-hand side of (2.4) converges to $\mathrm{E}[\exp(-t^2L/2); L < \infty]$.

We have proved that $f(t) = \mathrm{E}[\exp(-t^2L/2); L < \infty]$ for a possibly-degenerate non-negative random variable L. Set $t = 0$ to find that L is a proper random variable; i.e., $1 = f(0) = \mathrm{P}\{L < \infty\}$. Therefore, (2°) follows with ν denoting the distribution of L. □

References

Berg, Christian, Jens Peter Reus Christensen, and Paul Ressel. 1984. *Harmonic Analysis on Semigroups*, Graduate Texts in Mathematics, vol. 100, Springer-Verlag, New York.

Berg, Christian and Paul Ressel. 1978. *Une forme abstraite du théorème de Schoenberg*, Arch. Math. (Basel) **30***(1)*, 55–61. (French)

Bochner, Salomon. 1955. *Harmonic Analysis and the Theory of Probability*, University of California Press, Berkeley and Los Angeles.

Bretagnolle, J., D. Dacunha-Castelle, and J.-L. Krivine. 1967. *Lois stables et espaces L^p*, Symposium on Probability Methods in Analysis (Loutraki, 1966), Springer, Berlin, pp. 48–54. (French)

Bretagnolle, Jean, Didier Dacunha-Castelle, and Jean-Louis Krivine. 1965/1966. *Lois stables et espaces L^p*, Ann. Inst. H. Poincaré Sect. B (N.S.) **2**, 231–259. (French)

———. 1965. *Fonctions de type positif sur les espaces L^p*, C. R. Acad. Sci. Paris **261**, 2153–2156. (French)

Christensen, Jens Peter Reus and Paul Ressel. 1982. *Positive definite kernels on the complex Hilbert sphere*, Math. Z. **180***(2)*, 193–201.

Diaconis, Persi and David Freedman. 2004a. *The Markov moment problem and de Finetti's theorem. I*, Math. Z. **247***(1)*, 183–199.

———. 2004b. *The Markov moment problem and de Finetti's theorem. II*, Math. Z. **247***(1)*, 201–212.

Freedman, David A. 1963. *Invariants under mixing which generalize de Finetti's theorem: Continuous time parameter*, Ann. Math. Statist. **34**, 1194–1216.

Kahane, Jean-Pierre. 1985. *Some Random Series of Functions*, 2nd ed., Cambridge University Press, Cambridge.

Kelker, Douglas. 1970. *Distribution theory of spherical distributions and a location-scale parameter generalization*, Sankhyā Ser. A **32**, 419–438.

Kingman, J. F. C. 1972. *On random sequences with spherical symmetry*, Biometrika **59**, 492–494.

Koldobsky, Alexander. 1999. *Positive definite distributions and subspaces of L_{-p} with applications to stable processes*, Canad. Math. Bull. **42***(3)*, 344–353.

Koldobsky, Alexander and Yossi Lonke. 1999. *A short proof of Schoenberg's conjecture on positive definite functions*, Bull. London Math. Soc. **31***(6)*, 693–699.

Koldobsky, Alexander. 1996. *Positive definite functions, stable measures, and isometries on Banach spaces*, Interaction between Functional Analysis, Harmonic Analysis, and Probability (Columbia, MO, 1994), Lecture Notes in Pure and Appl. Math., vol. 175, Dekker, New York, pp. 275–290.

Misiewicz, Jolanta K. 1996a. *Substable and pseudo-isotropic processes—connections with the geometry of subspaces of L_α-spaces*, Dissertationes Math. (Rozprawy Mat.) **358**, 91.

———. 1996b. *Sub-stable and pseudo-isotropic processes—connections with the geometry of subspaces of L_α-spaces*, Bull. Polish Acad. Sci. Math. **44***(2)*, 209–235.

Ressel, Paul. 1985. *De Finetti-type theorems: an analytical approach*, Ann. Probab. **13***(3)*, 898–922.

———. 1976. *A short proof of Schoenberg's theorem*, Proc. Amer. Math. Soc. **57***(1)*, 66–68.

———. 1974. *Laplace-Transformation nichtnegativer und vektorwertiger Maße*, Manuscripta Math. **13**, 143–152. (German, with English summary)

Schoenberg, I. J. 1938. *Metric spaces and completely monotone functions*, Ann. of Math. (2) **39***(4)*, 811–841.

Steerneman, A. G. M.@ @@ and Frederieke van Perlo-ten Kleij. 2005. *Spherical distributions: Schoenberg (1938) revisited*, Expos. Math. (to appear).

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