

Gaussian Measure of a Small Ball and Capacity in Wiener Space

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Summary. We give asymptotic bounds for the Gaussian measure of a small ball in terms of the hitting probabilities of a suitably chosen infinite dimensional Brownian motion. Our estimates refine earlier works of Erickson [6].

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1. INTRODUCTION

Let $(\mathcal{B}, \|\cdot\|, \mu)$ denote a Gauss space in that $(\mathcal{B}, \|\cdot\|)$ is a Banach space which carries a centered, Gaussian measure μ living on the Borel σ -field of \mathcal{B} .

An old problem, which has received much recent attention, is to describe the rate at which the μ -measure of a small balanced ball (in \mathcal{B}) goes to zero, as the radius of the ball decreases to zero. To make this precise, suppose p is a continuous semi-norm on \mathcal{B} . That is, $p : \mathcal{B} \mapsto \mathbb{R}$ such that

- (i) For all $x, y \in \mathcal{B}$, $p(x + y) \leq p(x) + p(y)$;
- (ii) for all $\alpha \in \mathbb{R}$ and all $x \in \mathcal{B}$, $p(\alpha x) = |\alpha|p(x)$;
- (iii) whenever $x \rightarrow y$ in \mathcal{B} , $p(x - y) \rightarrow 0$.

In particular, note that $x \mapsto p(x)$ is a (nonnegative) continuous map from \mathcal{B} into \mathbb{R}_+ .

The so-called *small ball problem* for μ consists of finding a good approximation to the following:

$$\mu_p(r) \triangleq \mu(\omega \in \mathcal{B} : p(\omega) \leq r), \quad (1.1)$$

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as $r \rightarrow 0^+$. A major breakthrough is the recent work of Kuelbs and Li [11] relating μ_p to a combinatorial problem on the reproducing kernel Hilbert space corresponding to μ . The latter route typically leads one to long-standing open problems in functional analysis; cf. [11] for details.

Throughout this paper, we only deal with the case when p is transient. Rather than developing a theory for this, we define transience via the following technical assumption which will prevail throughout the rest of the paper:

Assumption 1.1. We assume the following two conditions:

- (a) p is *transient* in the sense that for some $\kappa > 2$,

$$\limsup_{r \rightarrow 0^+} r^{-\kappa} \mu_p(r) < \infty; \text{ and}$$

- (b) p is *nondegenerate* in the sense that $1 > \mu_p(1) > 0$.

Remark 1.1.1.

- (a) Suppose $\dim(\mathcal{B}) = d < \infty$. In other words, \mathcal{B} is finite dimensional with (topological) dimension d . It is easy to see that as $r \rightarrow 0^+$, $\mu_p(r) \sim Cr^d$ for some $C > 0$. Therefore, Assumption 1.1(a) says that $d > 2$ in this case. This corresponds to the well-known condition of transience in the classical sense.
- (b) When p has rank ≥ 3 , p is transient; see [6] for details.
- (c) If p is nondegenerate, there exists $c > 0$, such that $1 > \mu_p(c) > 0$. By considering the semi-norm $c^{-1}p$ instead, we see that Assumption 1.1(b) is not an essential restriction.

Motivated by the work of Erickson [6], this paper proposes a different approximation of μ_p . To begin, recall that the triple $(\mathcal{B}, \|\cdot\|, \mu)$ corresponds to a *Wiener space* C . To define it, let $C(\mathcal{B})$ denote the space of all continuous functions $\omega : [0, 1] \mapsto \mathcal{B}$ with $\omega(0) = 0$, endowed with the compact open topology. Let \mathcal{C} denote the associated Borel field. For all $\omega \in C(\mathcal{B})$, let $B_t(\omega) = \omega(t)$. It is a well known fact that there exists a probability measure \mathbb{P} on the measure space $(C(\mathcal{B}), \mathcal{C})$ which renders the process B a μ -Brownian motion; cf. Gross [8] or Üstünel [13] for a modern treatment as well as some of the new developments in this area. In particular, we mention the following important properties:

- (i) with \mathbb{P} -probability one, $t \mapsto B_t$ is continuous;
- (ii) B has independent and stationary increments (under \mathbb{P});
- (iii) for all $x \in \mathcal{B}^*$, the random variable $\langle x, B_t \rangle$ has a one-dimensional Gaussian distribution with mean 0 and variance $t \int_{\mathcal{B}} \|x\|^2 \mu(dx)$ (under \mathbb{P});
- (iv) B is a \mathcal{B} -valued diffusion.
- (v) $B_0 = 0$, \mathbb{P} -almost surely.

We will denote by \mathbb{E} the expectation operator corresponding to the underlying (Gaussian) probability measure \mathbb{P} .

It will be convenient to write our results in terms of the μ -Ornstein-Uhlenbeck process O given by

$$O_t \triangleq e^{-t/2} B_{e^t}, \quad t \geq 0. \quad (1.2)$$

We note in passing the elementary fact that O is a \mathcal{B} -valued stationary diffusion whose stationary measure is μ .

For any $\varkappa > 1$, define,

$$\lambda_p(r; \varkappa) \triangleq \sup \{a > 0 : \mu_p(a) \leq \varkappa \mu_p(r)\}, \quad r > 0. \quad (1.3)$$

In the finite-dimensional case, it is possible to show that for any $\varkappa > 0$, as $r \rightarrow 0^+$, $\lambda_p(r; \varkappa) \sim \varkappa^{1/d} r$, where $d \geq 3$ is the dimension of \mathcal{B} . In this connection, see also Remark 1.1.1(a).

The promised correspondence between μ_p and the process O can then be described in terms of λ_p as follows:

Theorem 1.2. *Suppose $p : \mathcal{B} \mapsto \mathbb{R}$ is a nondegenerate, transient semi-norm on \mathcal{B} . Then, for all $T > 0$, and all $\varkappa > 1$, there exists a constant $c \in (1, \infty)$ such that for all $r \in (0, 1/c)$,*

$$\frac{\mu_p(r)}{c r^2} \leq \mathbb{P}\left(\inf_{0 \leq t \leq T} p(O_t) \leq r\right) \leq \frac{c \mu_p(r)}{(\lambda_p(r; \varkappa) - r)^2}.$$

In fact, the proof of Theorem 1.2 can be used, with little change, to show the following:

Corollary 1.3. *Suppose $p : \mathcal{B} \rightarrow \mathbb{R}$ is a nondegenerate, transient semi-norm on \mathcal{B} . Then, for all $\lambda > 0$ and all $\varkappa > 1$, there exists a constant $c \in (1, \infty)$ such that for all $r \in (0, 1/c)$,*

$$\frac{\mu_p(r)}{c r^2} \leq \int_0^\infty e^{-\lambda T} \mathbb{P}\left(\inf_{0 \leq t \leq T} p(O_t) \leq r\right) dT \leq \frac{c \mu_p(r)}{(\lambda_p(r; \varkappa) - r)^2}.$$

Remark.

(a) The quantity,

$$\int_0^\infty e^{-\lambda T} \mathbb{P}\left(\inf_{0 \leq t \leq T} p(O_t) \leq r\right) dT,$$

is the λ -capacity of the “ball” $\{\omega \in \mathcal{B} : p(\omega) \leq r\}$; cf. Üstünel [13] and Fukushima et al. [7] for details. It turns out that under very general conditions, $\lambda_p(r; \varkappa) - r$ has polynomial decay rate; cf. Remark 1.1.1(a) above. Thus, Theorem 1.2 and its variants provide exact (and essentially equivalent) asymptotics between the λ -capacity of a small ball and the small ball probability μ_p given by (1.1).

- (b) In infinite dimensions – which is what is of interest here – the methods of Erickson [6] provide an upper bound of $\mu_p((1 + \varepsilon)r)$ for any $\varepsilon > 0$. In such cases, μ_p decays exponentially fast. Therefore, Theorem 1.2 is an essential improvement.

A refinement of Theorem 1.2 is proved in Section 2. Section 3 contains an explicit application. The methods of the latter section can be combined with those of [6] to provide a host of other examples.

2. THE MAIN ESTIMATE

In this section, we provide bounds for the probability that O hits a small ball in terms of the small ball probability μ_p defined by (1.1). The main result of this section is the following probability estimate:

Theorem 2.1. *Suppose p is a continuous, nondegenerate, transient semi-norm on \mathcal{B} in the sense of Assumption 1.1. Suppose further that $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a bounded measurable function satisfying: for some $\varkappa > 1$,*

$$I_g \triangleq \liminf_{r \rightarrow 0^+} \frac{g(r)}{(\lambda_p(r; \varkappa) - r)^2} > 0.$$

Then, there exists a constant $c \in (1, \infty)$ which depends only on \varkappa , $\sup_x g(x)$ and I_g , such that for all $r \in (0, 1/c)$,

$$\frac{\mu_p(r)g(r)}{c r^2} \leq \mathbb{P}\left(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r\right) \leq \frac{c \mu_p(r)g(r)}{(\lambda_p(r; \varkappa) - r)^2}.$$

Remark 2.1.1.

- (a) The special case when $g(r) \equiv T$ immediately yields Theorem 1.2.
 (b) Suppose $g \equiv T$ and $\dim(\mathcal{B}) = d < \infty$. This is the finite dimensional version of Theorem 2.1 and is well known. Namely, that in $d > 2$ dimensions, there exists a constant $c \in (1, \infty)$ such that for all r small enough,

$$c^{-1}r^{d-2} \leq \mathbb{P}\left(\inf_{0 \leq t \leq T} p(O_t) \leq r\right) \leq cr^{d-2}.$$

(Recall Remark 1.1.1(a) regarding transience in finite dimensions as well as the estimates for λ_p .)

- (c) The lower bound in Theorem 2.1 holds as long as g is bounded and measurable. More precisely, the condition that $I_g > 0$ is only needed for the upper bound.
 (d) According to Weber [14] (cf. also Albin [1] and Khoshnevisan and Shi [9]), there is a connection between the modulus of continuity of a Gaussian process and its

hitting probabilities. In our setting, the Gaussian process is infinite dimensional. In some infinite dimensional cases, such moduli of continuity are found; cf. Csáki and Csörgő [3] and Csáki et al. [4]. While it seems somewhat unlikely, one cannot help but ask if there is a connection between our results and such moduli of continuity in infinite dimensions.

The rest of this section is devoted to the proof of Theorem 2.1.

Lemma 2.2. *Define,*

$$W_t \triangleq \frac{O_t - e^{-t/2}O_0}{\sqrt{1 - e^{-t}}}. \quad (2.1)$$

For any fixed $t \geq 0$, W_t as an element of \mathcal{B} is an independent copy of O_0 , satisfying,

$$O_t = \sqrt{1 - e^{-t}}W_t + e^{-t/2}O_0.$$

Proof. By (1.2), $W_t = e^{-t/2}(B_{e^t} - B_1)$ which is independent of $O_0 = B_1$. To verify that W_t is distributed as O_0 , it suffices to show that for all $x \in \mathcal{B}^*$,

$$\langle x, O_t \rangle - e^{-t/2}\langle x, O_0 \rangle \stackrel{(d)}{=} \sqrt{1 - e^{-t}}\langle x, O_0 \rangle.$$

This is a finite dimensional result which can be readily verified by checking means and covariances. \diamond

Lemma 2.3. *Suppose $r > 0$ and $t \geq 0$ are fixed and p satisfies Assumption 1.1. Then uniformly over all $f \in \mathcal{B}$ with $p(f) \leq r$,*

$$\mathbb{P}(p(O_t) \leq r \mid O_0 = f) \leq \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right).$$

Proof. Let W_t be as in (2.1). By properties of p ,

$$\sqrt{1 - e^{-t}}p(W_t) \leq p(O_t) + e^{-t/2}p(O_0).$$

Therefore, conditional on $\{p(O_0) \leq r\}$, $\{p(O_t) \leq r\}$ implies $\{p(W_t) \leq cr\}$, where

$$c \triangleq \sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}}.$$

Since by Lemma 2.2 $p(W_t) \stackrel{(d)}{=} p(O_0)$, the result follows. \diamond

Lemma 2.4. *Suppose p is a continuous semi-norm on \mathcal{B} for which Assumption 1.1 is verified. Then for any $T > 0$,*

$$\limsup_{r \rightarrow 0^+} r^{-2} \int_0^T \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt < \infty.$$

Proof. Note that for all $0 < r < 1$,

$$\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \leq 1, \quad \text{if and only if} \quad t \geq 2 \ln \left(\frac{1 + r^2}{1 - r^2} \right).$$

Therefore, for all $r > 0$ small,

$$\int_0^T \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt \leq 2 \ln \left(\frac{1 + r^2}{1 - r^2} \right) + \int_{2 \ln \left(\frac{1 + r^2}{1 - r^2} \right)}^T \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt.$$

As $r \rightarrow 0^+$, the first term behaves like $4r^2$. On the other hand, for $t \leq T$,

$$\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} \leq \frac{c_1}{\sqrt{t}},$$

for some constant c_1 . By Assumption 1.1(a), there exist $c_2, c_3, c_4 > 0$, such that for all $r > 0$ small, the second term is bounded above by

$$c_2 \int_{c_3 r^2}^T \left(\frac{r}{\sqrt{t}} \right)^\kappa dt \leq c_4 r^2.$$

This concludes the proof. ◇

Lemma 2.4 is sharp. Indeed, using Assumption 1.1(b) instead of 1.1(a) in the proof of Lemma 2.4, we immediately arrive at the following:

Lemma 2.5. *Under the conditions of Lemma 2.4, for all $T > 0$,*

$$\liminf_{r \rightarrow 0^+} r^{-2} \int_0^T \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt > 0.$$

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Fix $\varkappa > 1$ as given. The conditions on g imply the existence of a constant $c \in (1, \infty)$, such that for all $r \in (0, 1)$,

$$c^{-1} (\lambda_p(r; \varkappa) - r)^2 \leq g(r) \leq c. \tag{2.2}$$

For all $r \geq 0$, define,

$$\tau(r) \triangleq \inf \{s > 0 : p(O_s) \leq r\}. \quad (2.3)$$

Since p is continuous, $p^{-1}([0, r])$ is closed. Therefore, $\tau(r)$ is a stopping time for the diffusion O . By (2.2) and the stationarity of O , for all $r \in (0, 1)$,

$$\begin{aligned} g(r)\mu_p(r) &= \mathbb{E} \left[\int_0^{g(r)} \mathbb{1}\{p(O_t) \leq r\} dt \right] \\ &= \mathbb{E} \left[\int_0^{g(r)} \mathbb{1}\{p(O_t) \leq r\} dt \mathbb{1}\{0 \leq \tau(r) < g(r)\} \right] \\ &\leq \mathbb{E} \left[\int_0^c \mathbb{1}\{p(O_t) \leq r\} dt \mathbb{1}\{0 \leq \tau(r) < g(r)\} \right], \end{aligned} \quad (2.4)$$

where $\mathbb{1}\{\dots\}$ is the indicator of whatever appears in the brackets. By Lemma 2.3,

$$\begin{aligned} \mathbb{P}(p(O_t) \leq r, \tau(r) = 0) &= \mathbb{P}(p(O_t) \leq r \mid p(O_0) \leq r) \mathbb{P}(\tau(r) = 0) \\ &\leq \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) \mathbb{P}(\tau(r) = 0). \end{aligned}$$

Therefore,

$$\mathbb{E} \left[\int_0^c \mathbb{1}\{p(O_t) \leq r\} dt \mathbb{1}\{\tau(r) = 0\} \right] \leq \mathbb{P}(\tau(r) = 0) \int_0^c \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt. \quad (2.5)$$

On the other hand, applying the strong Markov property at time $\tau(r)$, we arrive at the following:

$$\begin{aligned} &\mathbb{E} \left[\int_0^c \mathbb{1}\{p(O_t) \leq r\} dt \mathbb{1}\{0 < \tau(r) < g(r)\} \right] \\ &= \mathbb{E} \left[\mathbb{1}\{0 < \tau(r) < g(r)\} \int_0^{c-\tau(r)} \mathbb{1}\{p(O_t) \leq r\} dt \circ \theta(\tau(r)) \right] \\ &\leq \mathbb{P}(0 < \tau(r) < g(r)) \sup_f \mathbb{E} \left[\int_0^c \mathbb{1}\{p(O_t) \leq r\} dt \mid O_0 = f \right], \end{aligned}$$

where θ is the shift functional on the paths of the diffusion O and the supremum is over all $f \in \mathcal{B}$ with $p(f) \leq r$. By Lemma 2.3,

$$\begin{aligned} &\mathbb{E} \left[\int_0^c \mathbb{1}\{p(O_t) \leq r\} dt \mathbb{1}\{0 < \tau(r) < g(r)\} \right] \\ &\leq \mathbb{P}(0 < \tau(r) < g(r)) \cdot \int_0^c \mu_p \left(\sqrt{\frac{1 + e^{-t/2}}{1 - e^{-t/2}}} r \right) dt. \end{aligned}$$

Together with (2.5) and (2.4), this proves the following:

$$g(r)\mu_p(r) \leq \mathbb{P}\left(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r\right) \cdot \int_0^c \mu_p\left(\sqrt{\frac{1+e^{-t/2}}{1-e^{-t/2}}} r\right) dt.$$

By Lemma 2.4, for all $r > 0$ small, the right hand side is bounded above by a constant multiple of $r^2 \mathbb{P}(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r)$. In other words, we have proven the following:

$$\liminf_{r \rightarrow 0^+} \frac{r^2}{g(r)\mu_p(r)} \mathbb{P}\left(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r\right) > 0. \quad (2.6)$$

This constitutes the first half of Theorem 2.1. We have also verified Remark 2.1.1(c). The second half proceeds along different lines.

Recall the definition of λ_p from (1.3). Since μ_p is a decreasing function, one easily deduces that

- (i) $r \mapsto \lambda_p(r; \varkappa)$ is decreasing;
- (ii) $\lambda_p(r; \varkappa) \geq r$.

Fix $t \geq 0$. By Lemma 2.2 (in its notation), using the properties of p ,

$$p(O_t) \leq \sqrt{1 - e^{-t}} p(W_t) + e^{-t/2} p(O_0).$$

(This should be compared to the proof of Lemma 2.3). Observe that conditional on $\{p(O_0) \leq r\}$, $\{p(W_t) < 1\}$ implies the following:

$$p(O_t) \leq \sqrt{1 - e^{-t}} + r \leq t^{1/2} + r.$$

Therefore, for all $t \leq (\lambda_p(r; \varkappa) - r)^2$,

$$\inf_f \mathbb{P}(p(O_t) \leq \lambda_p(r; \varkappa) \mid O_0 = f) \geq \mathbb{P}(p(W_t) \leq 1) = \mu_p(1) > 0, \quad (2.7)$$

where the infimum is taken over all $f \in \mathcal{B}$ which satisfy $p(f) \leq r$. The rest of the proof follows from stationarity and the strong Markov property at time $\tau(r)$, viz.,

$$\begin{aligned} 3g(r)\mu_p(\lambda_p(r; \varkappa)) &= \mathbb{E}\left[\int_0^{3g(r)} \mathbb{1}\{p(O_t) \leq \lambda_p(r; \varkappa)\} dt\right] \\ &\geq \mathbb{E}\left[\int_{\tau(r)}^{3g(r)} \mathbb{1}\{p(O_t) \leq \lambda_p(r; \varkappa)\} dt \mid 0 \leq \tau(r) \leq g(r)\right] \cdot \mathbb{P}(0 \leq \tau(r) \leq g(r)) \\ &\geq \mathbb{E}\left[\int_0^{2g(r)} \mathbb{1}\{p(O_t) \leq \lambda_p(r; \varkappa)\} dt \circ \theta(\tau(r)) \mid 0 \leq \tau(r) \leq g(r)\right] \cdot \mathbb{P}(0 \leq \tau(r) \leq g(r)) \\ &\geq \inf_f \mathbb{E}\left[\int_0^{g(r)} \mathbb{1}\{p(O_t) \leq \lambda_p(r; \varkappa)\} dt \mid O_0 = f\right] \cdot \mathbb{P}(0 \leq \tau(r) \leq g(r)), \end{aligned}$$

where the infimum is taken over all $f \in \mathcal{B}$ such that $p(f) \leq r$ and θ is as before the shift on the paths of O . By (2.2),

$$\begin{aligned} & 3g(r)\mu_p(\lambda_p(r; \varkappa)) \\ & \geq \inf_f \mathbb{E} \left[\int_0^{c^{-1}(\lambda_p(r; \varkappa) - r)^2} \mathbb{1}\{p(O_t) \leq r\} dt \mid O_0 = f \right] \mathbb{P}(0 \leq \tau(r) \leq g(r)), \end{aligned}$$

where the infimum is taken over all $f \in \mathcal{B}$ with $p(f) \leq r$. Since $c > 1$, (2.7) implies

$$\begin{aligned} 3g(r)\mu_p(\lambda_p(r; \varkappa)) & \geq \mu_p(1) c^{-1}(\lambda_p(r; \varkappa) - r)^2 \mathbb{P}(0 \leq \tau(r) \leq g(r)) \\ & = \mu_p(1) c^{-1}(\lambda_p(r; \varkappa) - r)^2 \mathbb{P}\left(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r\right). \end{aligned}$$

By Assumption 1.1(b), $\mu_p(1) > 0$. By (1.3), we have proven the following:

$$\mathbb{P}\left(\inf_{0 \leq t \leq g(r)} p(O_t) \leq r\right) \leq \frac{3c\varkappa}{\mu_p(1)} \frac{\mu_p(r)g(r)}{(\lambda_p(r; \varkappa) - r)^2}.$$

Together with (2.6), this proves Theorem 2.1. ◇

3. AN APPLICATION

Recall the μ -Brownian motion $B = (B_t; t \geq 0)$ from Introduction. The goal of this section is to provide some estimates of the escape rates of B when p is transient. In light of Remark 1.1.1, it is not too difficult to convince oneself that under Assumption 1.1, $t \mapsto p(B_t)$ is transient in that \mathbb{P} -a.s., $\lim_{t \rightarrow \infty} p(B_t) = \infty$. It is the goal of this section to estimate the rate at which this blow-up occurs. We improve results of Erickson [6] concerning the following class of problems.

Assumption 3.1. We assume the existence of constants $r_0 > 0$, $K_0 > 1$, $0 < \alpha \leq 2$, $\chi > 0$ and $\beta \in \mathbb{R}$ such that for all $0 < r \leq r_0$,

$$K_0^{-1}r^\beta \exp\left(-\frac{\chi}{r^\alpha}\right) \leq \mu_p(r) \leq K_0r^\beta \exp\left(-\frac{\chi}{r^\alpha}\right). \quad (3.1)$$

See Erickson [6], Li [12] and their combined references for many examples of when this assumption is valid.

Note that under Assumption 3.1, \mathcal{B} is forced to be infinite-dimensional; see Remark 1.1.1. Moreover, Assumption 3.1 implies Assumption 1.1.

Our result on escape rates is the following:

Theorem 3.2. *Suppose Assumption 3.1 is satisfied. Consider a measurable nonincreasing function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Let*

$$I_1(\psi) \triangleq \int_1^\infty \psi^{\beta-\alpha}(t) \exp\left(-\frac{\chi}{\psi^\alpha(t)}\right) \frac{dt}{t},$$

$$I_2(\psi) \triangleq \int_1^\infty \psi^{\beta-2\alpha-2}(t) \exp\left(-\frac{\chi}{\psi^\alpha(t)}\right) \frac{dt}{t}.$$

Then

$$I_1(\psi) = \infty \implies \mathbb{P}(p(B_t) \geq t^{1/2}\psi(t), \text{ eventually for all } t \text{ large}) = 0, \quad (3.2)$$

$$I_2(\psi) < \infty \implies \mathbb{P}(p(B_t) \geq t^{1/2}\psi(t), \text{ eventually for all } t \text{ large}) = 1. \quad (3.3)$$

Proof of (3.2). Assume $I_1(\psi) = \infty$. Define the Erdős sequence, $t_k \triangleq \exp(k/\log k)$. For simplicity, write $\psi_k \triangleq \psi(t_k)$. Furthermore, define the measurable event,

$$E_k \triangleq \{p(B_{t_k}) \leq t_k^{1/2}\psi_k\}.$$

According to an argument of Erdős [5], there exists a constant $C > 1$ such that,

$$C^{-1} (\log \log t_k)^{-1/\alpha} \leq \psi_k \leq C (\log \log t_k)^{-1/\alpha}. \quad (3.4)$$

By Brownian scaling,

$$\mathbb{P}(E_k) = \mu_p(\psi_k).$$

Since $I_1(\psi) = \infty$, (3.4) and Assumption 3.1 together yield:

$$\sum_k \mathbb{P}(E_k) = \infty.$$

Suppose we could show

$$\limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N \sum_{n=1}^{N-k} \mathbb{P}(E_k \cap E_{n+k})}{\left(\sum_{k=1}^N \mathbb{P}(E_k)\right)^2} < \infty, \quad (3.5)$$

then, according to Kochen and Stone [10], $\mathbb{P}(E_k, \text{ infinitely often}) > 0$. The latter is a tail event. By the classical 0–1 law of Kolmogorov, B_t has a trivial σ -field. Therefore, (3.5) implies $\mathbb{P}(E_k, \text{ i.o.}) = 1$, which yields (3.2).

It remains to prove (3.5). By Anderson's inequality (cf. [2]) and elementary Brownian properties (cf. Introduction),

$$\begin{aligned} \mathbb{P}(E_k \cap E_{n+k}) &= \mathbb{P}(E_k; p(B_{t_{n+k}} - B_{t_k} + B_{t_k}) \leq t_{n+k}^{1/2}\psi_{n+k}) \\ &\leq \mathbb{P}(E_k) \mathbb{P}(p(B_{t_{n+k}} - B_{t_k}) \leq t_{n+k}^{1/2}\psi_{n+k}) \\ &= \mathbb{P}(E_k) \mu_p\left(\sqrt{\frac{t_{n+k}}{t_{n+k} - t_k}}\psi_{n+k}\right). \end{aligned}$$

From here, we can apply a classical argument going back at least to Erdős [5], and only provide its outline. Let C_i ($1 \leq i \leq 9$) denote some unimportant constants. When $n \leq \log k$, $\sqrt{t_{n+k}/(t_{n+k} - t_k)}\psi_{n+k} \leq C_1 n^{-1/2}$, which implies

$$\mathbb{P}(E_k \cap E_{n+k}) \leq C_2 \mathbb{P}(E_k) \exp(-C_3 n^{\alpha/2}). \quad (3.6)$$

For $\log k < n < (\log k)^2$, we have $t_{n+k}/(t_{n+k} - t_k) \leq C_4$, which yields

$$\mathbb{P}(E_k \cap E_{n+k}) \leq C_5 \mathbb{P}(E_k) \exp(-C_6 \log k) \leq C_5 \mathbb{P}(E_k) \exp(-C_6 \sqrt{n}). \quad (3.7)$$

Finally, if $n \geq (\log k)^2$, then $t_{n+k}/(t_{n+k} - t_k) \leq 1 + C_7 (\log(n+k))/n$. Hence

$$\begin{aligned} \mathbb{P}(E_k \cap E_{n+k}) &\leq C_8 \mathbb{P}(E_k) \psi_{n+k}^\beta \exp\left(-\frac{\chi}{\psi_{n+k}^\alpha}\right) \\ &\leq C_9 \mathbb{P}(E_k) \mathbb{P}(E_{n+k}). \end{aligned} \quad (3.8)$$

Combining (3.6)–(3.8) gives (3.5). ◇

To prove the other part of Theorem 3.2, we need a preliminary result.

Lemma 3.3. *Under Assumption 3.1, there exist $\varkappa > 1$, $r_1 > 0$ and $K_1 > 1$ such that for all $0 < r \leq r_1$,*

$$r + K_1^{-1} r^{1+\alpha} \leq \lambda_p(r; \varkappa) \leq r + K_1 r^{1+\alpha}.$$

Proof. We will prove the upper bound for $\lambda_p(r; \varkappa)$. The lower bound follows from similar arguments. Fix any $\gamma > 0$. Two applications of (3.1) show that for all $0 < r \leq r_0$ so small that $r + \gamma r^{1+\alpha} \leq r_0$,

$$\begin{aligned} \mu_p(r + \gamma r^{1+\alpha}) &\leq K_0 (r + \gamma r^{1+\alpha})^\beta \exp\left(-\frac{\chi}{(r + \gamma r^{1+\alpha})^\alpha}\right) \\ &\leq K_0^2 [(1 + \gamma r^\alpha)^\beta \vee 1] \mu_p(r) \mathcal{Q} \end{aligned}$$

where,

$$\mathcal{Q} \triangleq \exp\left(\frac{\chi}{r^\alpha} - \frac{\chi}{(r + \gamma r^{1+\alpha})^\alpha}\right).$$

A little calculus shows that whenever $\alpha > 0$, for all $0 < r \leq r_0$,

$$1 - \frac{1}{(1 + \gamma r^\alpha)^\alpha} \leq ((1 + \gamma r_0)^{\alpha-1} \vee 1) \alpha \gamma r^\alpha.$$

Hence,

$$\mathcal{Q} \leq \exp\left(\chi \alpha \gamma (1 \vee (1 + \gamma r_0)^{\alpha-1})\right).$$

Fix any $\varkappa > K_0^2$. Note that for all $\gamma > 0$ small,

$$K_0^2 [(1 + \gamma r_0^\alpha)^\beta \vee 1] \exp\left(\chi \alpha \gamma [1 \vee (1 + \gamma r_0)^\alpha]\right) \leq \varkappa.$$

Thus, we have shown that $\mu_p(r + \gamma r^{1+\alpha}) \leq \varkappa \mu_p(r)$. That is, for any $\varkappa > K_0^2$ and all $\gamma > 0$ small enough, $\lambda_p(r; \varkappa) \geq r + \gamma r^{1+\alpha}$. \diamond

Corollary 3.4. *Suppose that Assumption 3.1 holds. For $K > 0$, there exists a constant $c > 1$ such that for all $r \in (0, 1/c)$,*

$$\begin{aligned} c^{-1} r^{\beta+\alpha-2} \exp\left(-\frac{\chi}{r^\alpha}\right) &\leq \mathbb{P}(p(B_t) \leq t^{1/2}r, \text{ for some } t \in [1, 1 + K r^\alpha]) \\ &\leq c r^{\beta-\alpha-2} \exp\left(-\frac{\chi}{r^\alpha}\right). \end{aligned}$$

We are now ready to complete the proof of Theorem 3.2.

Proof of (3.3). Assume $I_2(\psi) < \infty$. Let t_k and ψ_k be as in the proof of (3.2). Define,

$$\mathcal{U}_k \triangleq \mathbb{P}(p(B_t) \leq t_{k+1}^{1/2} \psi_k, \text{ for some } t \in [t_k, t_{k+1}]).$$

By Brownian scaling,

$$\mathcal{U}_k \leq \mathbb{P}(p(B_t) \leq t^{1/2} \varphi_k, \text{ for some } t \in [1, t_{k+1}/t_k]),$$

where $\varphi_k \triangleq (t_{k+1}/t_k)^{1/2} \psi_k$. By (3.4), $t_{k+1}/t_k = 1 + O(\varphi_k^\alpha)$. Therefore, Corollary 3.4 shows that

$$\mathcal{U}_k \leq C_9 \varphi_k^{\beta-\alpha-2} \exp\left(-\frac{\chi}{\varphi_k^\alpha}\right) \leq C_{10} \psi_k^{\beta-\alpha-2} \exp\left(-\frac{\chi}{\psi_k^\alpha}\right),$$

where C_9 and C_{10} are two constants. Since $I_2(\psi) < \infty$, this yields $\sum_k \mathcal{U}_k < \infty$. By the Borel–Cantelli lemma, for all k_0 large enough (random but finite \mathbb{P} -a.s.), the following holds \mathbb{P} -a.s. for all $k \geq k_0$,

$$\inf_{t_k \leq t \leq t_{k+1}} p(B_t) \geq t_{k+1}^{1/2} \psi_k.$$

Take any $k \geq k_0$ and $t \in [t_k, t_{k+1}]$. Then, by monotonicity,

$$p(B_t) \geq t^{1/2} \psi(t),$$

as desired. \diamond

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