Notes on Continuity

Definition: A function f is continuous at $x = c$ if all of the following conditions are satisfied:
(i) $f(c)$ exists, i.e., $f(x)$ has a definite value at $x = c$.
(ii) $\lim_{x \to c} f(x)$ exists.
(iii) $\lim_{x \to c} f(x) = f(c)$.
If one or more of the conditions above do not hold, we say the function is **discontinuous at** $x = c$.

- Every **polynomial** function is **continuous at all points**.
- A rational function $\frac{f(x)}{g(x)}$ is **discontinuous for those values of** x **where** $g(x) = 0$ and **continuous at all other points**.
- To find the **points of discontinuity** of a rational function $\frac{f(x)}{g(x)}$ solve the equation $g(x) = 0$. The solutions of this equation are *the* points of discontinuity. Any other point which is not a solution of the equation $g(x) = 0$, $\frac{f(x)}{g(x)}$ is continuous there.
- **Examples:** Find the points of discontinuity of the the following functions:
  1. $\dfrac{x^3 - 1}{x^2 - x - 6}$
  2. $\dfrac{x^2}{x}$
  3. $\dfrac{x}{x^2 + 1}$
- The points of discontinuity of (1) are solutions of $x^2 - x - 6 = 0$, which are $x = -2, 3$.
  The same for (2) is the solution of $x = 0$, i.e. 0 is the only point of discontinuity of the function in (2).
  For the function in (3), discontinuities are solution of $x^2 + 1 = 0$, but this equation **does not have any real root**, so this function is **continuous everywhere**. This example (3) shows that **polynomials are Not the only functions which are continuous everywhere**, there are some more.

- For a piecewise defined function $f(x)$, **if each of its pieces is defined by a polynomial** then the function **may or may not be continuous at the points where $f(x)$ changes its formula**, but it is continuous at all other points.

- **Example:** $f(x) = \begin{cases} x^2 - x + 1 & \text{if } x \leq 0 \\ 2x^3 + 1 & \text{if } 0 < x < 1 \\ 4 & \text{if } x \geq 1 \end{cases}$

  The function is changing its formula at 0 and 1, so the only possible discontinuities of $f(x)$ are at 0 and 1, the function is continuous at all other points.

  Now, $f(0) = 0^2 - 0 + 1 = 1$, so $f(0)$ exists.
  $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 - x + 1 = 0^2 - 0 + 1 = 1$
  $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x^3 + 1 = 2 \cdot 0^3 + 1 = 1$
  Therefore $\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x)$, so, $\lim_{x \to 0} f(x)$ exists and
  $\lim_{x \to 0} f(x) = 1 = f(0)$
  Hence $f(x)$ is continuous at $x = 0$.

  Now look at $x = 1$, $f(1) = 4$, $f(1)$ exists.
  $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x^3 + 1 = 3$
  $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4 = 4$
  $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$, so $\lim_{x \to 1} f(x)$ does not exists.
  Hence $f(x)$ is **Not** continuous at $x = 1$ (even though $f(1)$ is defined, $f(x)$ fails to satisfy the condition (ii) of the definition).
  Therefore the only point of discontinuity of $f(x)$ is $x = 1$, at all other points the function is continuous.