Problems 1-12, verify by substitution that each function is a solution of the given differential equation.

2) $y' + 2y = 0 \quad ; \quad y = 3e^{-2x}$

soln: Start by taking the derivative... $y' = -6e^{-2x}$
Then plug back into equation..
$y' + 2y = -6e^{-2x} + 2\cdot 3e^{-2x} = 0$ ✓

3) $y'' + 4y = 0 \quad ; \quad y_1 = \cos 2x \quad , \quad y_2 = \sin 2x$

soln: $y_1' = -2\sin 2x, \quad y_1'' = -4\cos 2x, \quad y_2' = 2\cos 2x, \quad y_2'' = -4\sin 2x$

$y_1'' + 4y_1 = -4\cos 2x + 4\cos 2x = 0$ ✓
$y_2'' + 4y_2 = -4\sin 2x + 4\sin 2x = 0$ ✓

4) $y'' = 9y \quad ; \quad y_1 = e^{3x} \quad , \quad y_2 = e^{-3x}$

soln: $y_1' = 3e^{3x}, \quad y_1'' = 9e^{3x} = 9y_1 \Rightarrow y_1'' = 9y_1$ ✓
$y_2'' = -3e^{-3x}, \quad y_2'' = 9e^{-3x} = 9y_2 \Rightarrow y_2'' = 9y_2$ ✓

7) $y'' - 2y' + 2y = 0 \quad ; \quad y_1 = e^x \cos x \quad , \quad y_2 = e^x \sin x$

soln: $y_1' = -e^x \sin x + e^x \cos x, \quad y_1'' = -2e^x \sin x$
$y_2' = e^x \sin x + e^x \cos x, \quad y_2'' = 2e^x \cos x$

so $y_1'' - 2y_1' + 2y_1 = -2e^x \sin x - 2(e^x(\cos x - \sin x)) + 2e^x \cos x = 0$ ✓
$y_2'' - 2y_2' + 2y_2 = 2e^x \cos x - 2(e^x(\cos x + \sin x)) + 2e^x \sin x = 0$ ✓

10) $x^2 y'' + xy' - y = \ln x \quad ; \quad y_1 = x - \ln x \quad , \quad y_2 = \frac{1}{x} - \ln x$

soln: $y_1' = 1 - \frac{1}{x}, \quad y_1'' = \frac{1}{x^2}, \quad y_2' = -\frac{1}{x^2} - \frac{1}{x}, \quad y_2'' = \frac{2}{x^3} + \frac{1}{x^2}$

so $x^2 y_1'' + xy_1' - y_1 = x^2\left(\frac{1}{x^2}\right) + x\left(1 - \frac{1}{x}\right) - (x - \ln x)$
$= 1 + x - 1 - x + \ln x = \ln x$ ✓

and $x^2 y_2'' + xy_2' - y_2 = x^2\left(\frac{2}{x^3} + \frac{1}{x^2}\right) + x\left(-\frac{1}{x^2} - \frac{1}{x}\right) - \left(\frac{1}{x} - \ln x\right)$
$= 2/x + 1 - 1/x - 1 - 1/x + \ln x$
$= \ln x$ ✓

12) $x^2 y'' - xy' + 2y = 0 \quad ; \quad y_1 = x\cos(\ln x), \quad y_2 = x\sin(\ln x)$

soln: $y_1' = x\cdot(\cos(\ln x))' + \cos(\ln x) = x\cdot(-\sin(\ln x)\cdot\frac{1}{x}) + \cos \ln(x)$
$= \cos(\ln x) - \sin(\ln x)$
$y_1'' = -\sin(\ln x)\cdot\frac{1}{x} - \cos(\ln x)\cdot\frac{1}{x} = -\frac{1}{x}(\sin(\ln x) + \cos(\ln x))$

$x^2\left(-\frac{1}{x}(\sin(\ln x) + \cos(\ln x))\right) - x(\cos(\ln x) - \sin(\ln x)) + 2(x\cos(\ln x))$
$= -x\sin(\ln x) - x\cos(\ln x) - x\cos(\ln x) + x\sin(\ln x) + 2x\cos(\ln x)$
$-2x\cos(\ln x) + 2x\cos(\ln x) = 0$ ✓

Problems 13-16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant $r$ for which $y = e^{rx}$ is a solution.

15) $y'' + y' - 2y = 0$

soln: $y' = re^{rx}$ , $y'' = r^2e^{rx}$ ← Plug these into eqn.

$$r^2e^{rx} + re^{rx} - 2e^{rx} = 0 \Rightarrow r^2 + r - 2 = 0$$

which simplifies to $(r+2)(r-1) = 0$.

$\therefore$ $r_1 = -2$ , $r_2 = 1$ .

16) $3y'' + 3y' - 4y = 0$

soln: $3r^2e^{rx} + 3re^{rx} - 4e^{rx} = 0 \Rightarrow 3r^2 + 3r - 4 = 0$

using $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow \dfrac{-3 \pm \sqrt{9 + 48}}{6}$

$\therefore$ $r_1 = \dfrac{-3 + \sqrt{57}}{6}$ , $r_2 = \dfrac{-3 - \sqrt{57}}{6}$ .

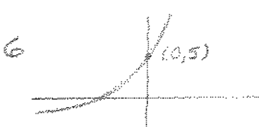
Problems 17-26. first verify that $y(x)$ satisfies the given diff. eqn. Then determine a value of the constant $C$ such that $y(x)$ satisfies the I.C.

19) $y' = y + 1$ , $y(x) = Ce^x - 1$ , $y(0) = 5$

soln: Check → $y'(x) = Ce^x \Rightarrow Ce^x = (Ce^x - 1) + 1 = Ce^x$ ✓

Now I.C. → $y(0) = 5 = Ce^0 - 1 = 5$

$= C - 1 = 5 \Rightarrow C = 6$




Therefore, the solution to this initial value problem is $\underline{y(x) = 6e^x - 1}$

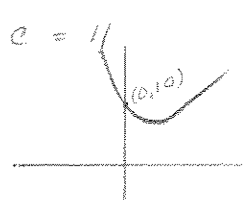
20) $y' = x - y$ ; $y(x) = Ce^{-x} + x - 1$ , $y(0) = 10$

soln: Check → $y'(x) = -Ce^{-x} + 1$

$-Ce^{-x} + 1 = x - Ce^{-x} - \cancel{x} + 1 = -Ce^{-x} + 1$ ✓

Now I.C. → $y(0) = 10$

$y(0) = C \cdot e^0 - 1 = C - 1 = 10 \Rightarrow C = 11$




Therefore, the solution to this initial value problem is:

$\underline{y(x) = 11e^{-x} + x - 1}$

Problems 27-31, write a D.E. of the form $dy/dx = f(x,y)$ having $g$ as soln.

27) The slope of the graph of $g$ at the point $(x, y)$ is the sum of $x$ and $y$.

$x + y$

soln: Recall $y = g(x)$
(solution is a function of $x$)
so the slope of $g \Rightarrow y'$ $\Rightarrow y' = x + y$ ✓

31) The line tangent to the graph of $g$ at $(x,y)$ passes through the point $(-y, x)$. (note the switch!)

soln: First find the slope of the line through $(x,y)$ and $(-y,x)$.

$$y' = \frac{x-y}{-y-x} = \frac{-(x-y)}{x+y} = \frac{y-x}{x+y}$$

which makes the D.E. $(x+y)y' = y-x$.

Problems 32-36, write a differential eqn. that is a mathematical model of the situation described.

34) The acceleration $dv/dt$ of a Lamborghini is proportional to the difference between 250 km/h and the velocity of the car.

soln: $$\frac{dv}{dt} = K(250 - v)$$

36) In a city with a fixed population of $P$ persons the time rate of change of the number $N$ of those persons infected with a certain contagious disease is proportional to the product of the number who have the disease and the number who do not.

soln: $$\frac{dN}{dt} = K(N)(P-N)$$

46) Suppose the velocity $v$ of a motorboat coasting in water satisfies the differential equation $dv/dt = kv^2$. The initial speed of the motorboat is $v(0) = 10$ m/s, and $v$ is decreasing at the rate of $1\ m/s^2$ when $v = 5$ m/s. How long does it take for the velocity of the boat to decrease to 1 m/s? To 1/10 m/s? When does the boat come to a stop?

soln: We are given $v' = kv^2$, $\underbrace{v' = -1, \quad v = 5}_{\text{plug in}}$.

$$-1 = k \cdot 25 \Rightarrow k = -1/25.$$

$$v' = -\tfrac{1}{25}v^2.$$

Look at problem 43 a). This tells us that the general (1-parameter) solution of a D.E. of this form is $v(t) = 1/(c + t/25)$

Now use the initial condition: $v(0) = \dfrac{1}{c + 0/25} = 10 \Rightarrow c = \dfrac{1}{10}$

$$\text{so } v(t) = \frac{1}{1/10 + t/25} = \frac{50}{5+2t}$$

when $v(t) = 1$, $t = 22.5$

when $v(t) = 1/10$, $t = 247.5$

And $v(t) \to 0$ as $t \to$ infinity.
So the boat never comes to a full stop in a finite period of time.

Problems 1-10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition

2) $\frac{dy}{dx} = (x-2)^2$ , $y(2) = 1$

soln: $y(x) = \int (x-2)^2 dx = \frac{1}{3}(x-2)^3 + C$
$y(2) = \frac{1}{3}(2-2)^3 + C = 1 \Rightarrow C = 1$
$\Rightarrow \underline{y(x) = \frac{1}{3}(x-2)^3 + 1}$

8) $\frac{dy}{dx} = \cos(2x)$ ; $y(0) = 1$

soln. $y(x) = \int \cos(2x)\,dx = \frac{1}{2}\sin 2x + C$
$y(0) = \frac{1}{2}\sin(0) + C = 1 \Rightarrow C = 1$
$\Rightarrow \underline{y(x) = \frac{1}{2}\sin(2x) + 1}$

10) $\frac{dy}{dx} = xe^{-x}$ ; $y(0) = 1$

soln: $y(x) = \int xe^{-x}\,dx$ , use u-substitution! (let $-x = u$ ; $-dx = du$ )
$= \int ue^{u}\,du$
From Formula #46 in back of book,
$= (u-1)e^{u} + C$
$-x = u$
$= -(x+1)e^{-x} + C$
Now $y(0) = -(1)e^{0} + C = 1 \Rightarrow -1 + C = 1 \Rightarrow C = 2$
$\Rightarrow \underline{y(x) = -(x+1)e^{-x} + 2}$

Problems 11-18, find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x_0 = x(0)$, and $v_0 = v(0)$.

15) $a(t) = 4(t+3)^2$ , $v_0 = -1$ , $x_0 = 1$

soln: Since $\frac{dv(t)}{dt} = a(t)$, $v(t) = \int a(t)\,dt$
$v(t) = \int 4(t+3)^2\,dt = \frac{4}{3}(t+3)^3 + C_1$
$v(0) = \frac{4}{3}(3)^3 + C_1 = -1 \Rightarrow C_1 = -37$

$\underline{v(t) = \frac{4}{3}(t+3)^3 - 37}$.

Now since $\frac{dx(t)}{dt} = v(t)$ , $x(t) = \int v(t)\,dt$
$x(t) = \int (\frac{4}{3}(t+3)^3 - 37)\,dt$
$= \frac{1}{3}(t+3)^4 - 37t + C_2$

$x(0) = \frac{1}{3}(3)^4 + C_2 = 1 \Rightarrow C_2 = -26$

$\Rightarrow \underline{x(t) = \frac{1}{3}(t+3)^4 - 37t - 26}$

16) $a(t) = \dfrac{1}{\sqrt{t+4}}$, $v_0 = -1$, $x_0 = 1$

soln: $v(t) = \int \frac{1}{\sqrt{t+4}}\,dt = 2\sqrt{t+4} + C_1$

$v(0) = 2\sqrt{0+4} + C_1 = -1 \Rightarrow C_1 = -5$

$\Rightarrow v(t) = 2\sqrt{t+4} - 5$

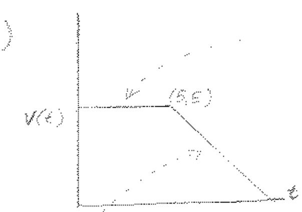
$x(t) = \int (2\sqrt{t+4} - 5)\,dt = \frac{4}{3}(t+4)^{3/2} - 5t + C_2$

$x(0) = \frac{4}{3}(4)^{3/2} + C_2 = 1 \Rightarrow C_2 = -29/3$

$\Rightarrow x(t) = \frac{4}{3}(t+4)^{3/2} - 5t - 29/3$

Problems 19-21 a particle starts at the origin(*) and travels along the x-axis with the velocity function v(t). Sketch its position function.

19)



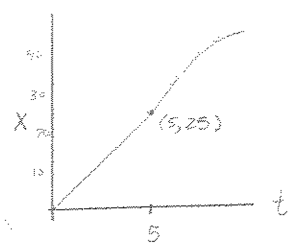

$v = 5$ for $0 \le t \le 5$

$\Rightarrow x(t) = 5t + C_1$

*) $x(0) = 0$

$x(0) = C_1 = 0 \Rightarrow C_1 = 0$

$x(t) = 5t$ // ← plot




$v = 10 - t$ for $5 \le t \le 10$

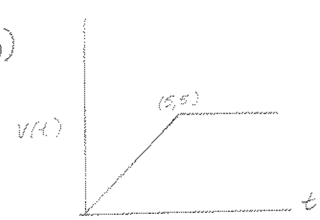
$x(t) = 10t - \frac{1}{2}t^2 + C_2$

I.C. is more of a continuity condition for $5 \le t \le 10$.

when $t = 5$, $10(5) - \frac{1}{2}(5)^2 + C_2 = 5(5) \Rightarrow C_2 = -25/2$

$x(t) = 10t - \frac{1}{2}t^2 - 25/2$ // ←

20)




Now for $0 \le t \le 5$, $v = t$

That means $x = \frac{1}{2}t^2 + C$

$x(0) = C = 0$.

$\therefore x(t) = \frac{1}{2}t^2$ for $0 \le t \le 5$.

For $5 \le t \le 10$, $v = 5$.

Therefore $x(t) = 5t + C$

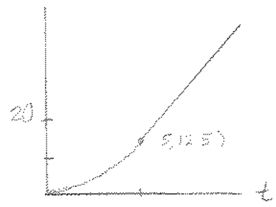
and continuity requires that

$\frac{1}{2}t^2$ and $5t + C$ agree when $t = 5$.

$\Rightarrow \frac{1}{2}(5)^2 = 5(5) + C$

$25/2 = 25 + C \Rightarrow C = -25/2$.

$\therefore x(t) = 5t - 25/2$ for $5 \le t \le 10$



28) A baseball is thrown straight downward with an initial speed of 40 ft/s from the top of Washington Monument (555 ft high). How long does it take to reach the ground and with what speed does the ball strike the ground?

soln. We are in ft/s, so use fps units (p. 13)

$g = 32\ ft/s^2$ — plug this into eqn 16), $v(t) = -gt + v_0$ with $v_0 = 40$ ft/s.

$$v(t) = -32t + 40$$

now from eqn 17), $-\frac{1}{2}gt^2 + v_0 t + y_0 = y(t)$ we get

$$y(t) = -\tfrac{1}{2}32 \cdot t^2 + 40t + 555$$

The ball hits the ground when $y = 0$.

This happens at $t = 4.77$ seconds

velocity $= v(4.77) = -192.64$ ft/sec. //

44) A driver involved in an accident claims he was going only 25 mph. When police tested his car, they found that when its brakes were applied at 25 mph, the car skidded only 45 ft before coming to a stop. But the driver's skidmarks at the accident measured 210 ft. Assuming the same (constant) deceleration, determine the speed he was actually travelling

soln: → Let $x_0$ be the initial position (at $t = 0$) when the brakes are applied (i.e. $x_0 = 0$).

→ $x''(t) = -a$ (we are decelerating)

$x'(t) = -at + v_0$, $x(t) = -\frac{1}{2}at^2 + v_0 t + 0$

Police find that ..

$v_0 = 25$, $x'(t) = 0$ when $x(t) = 45$.

so $x'(t) = 0 = -at + 25 \Rightarrow t = 25/a$;

$x(25/a) = 45$

$\Rightarrow -\frac{1}{2}a\left(\frac{25}{a}\right)^2 + 25\left(\frac{25}{a}\right) = 45$

$\Rightarrow -\frac{1}{2a}(25)^2 + \frac{1}{a}(25)^2 = \frac{1}{2a}(25)^2 = 45 \Rightarrow \underline{a = \frac{125}{18}}$

So now what is $v_0$ if $x'(t) = 0$ when $x(t) = 210$?

$$210 = -\tfrac{1}{2}at^2 + v_0 t$$

and

$$x'(t) = -at + v_0 = 0 \Rightarrow t = v_0/a$$

$$210 = -\tfrac{1}{2}a\left(\tfrac{v_0}{a}\right)^2 + v_0\left(\tfrac{v_0}{a}\right) = \tfrac{1}{2a}v_0^2$$

$$210 \cdot 2a = v_0^2 = 420 \cdot a = 420\,\tfrac{125}{18}$$

$$v_0 = \sqrt{\frac{420 \cdot 125}{18}} = 54 \text{ miles / hour.} //$$