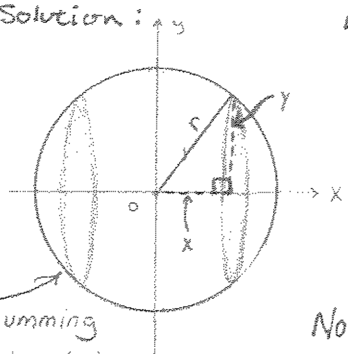
§ 8.1 | AREAS & VOLUMES | QUIZ # 3 REVIEW

$\boxed{\text{EX}}$ Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution:



We are summing up all the disks in the vertical direction. (We could have easily done this horizontally... ASK YOURSELF: What would change?)

Let's approach this problem by integrating disks along the x-axis.

$$y = \sqrt{r^2 - x^2}$$

The cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Now find volume by integrating

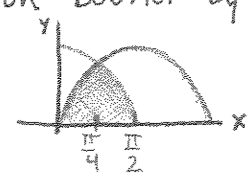
$$V = \int_{-r}^{r} A(x)dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= 2\pi \int_0^r (r^2 - x^2)\,dx$$

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = 2\pi\left[r^3 - \frac{r^3}{3}\right] = \frac{4}{3}\pi r^3$$

$\boxed{\text{EX}}$ (HW # 24) (kind of...) $y = \sin(x)$, $y = \cos(x)$, $y = 0$

The region bound by these curves:



1) • Find the volume of the solid obtained by rotating this region about the x-axis.

Solution:

Let's break this up into 2 regions - 0 to $\pi/4$, the "sin(x)" side, and $\pi/4$ to $\pi/2$, the "cos(x)" side.

• Evaluate by "vertical slices" along x-axis

$\Rightarrow \int A(x)\,dx$

where $A(x) = \pi y(x)^2$

$$V = \int_0^{\pi/4} \pi \sin(x)^2 dx + \int_{\pi/4}^{\pi/2} \pi \cos(x)^2 dx$$

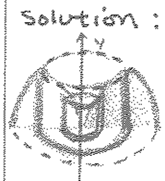
(HINT: Probably a good idea to know how to find these integrals by now...)

YOU CONFIRM:

$$V = \frac{\pi^2}{4} - \frac{\pi}{2}$$

2) • Find the volume of the solid obtained by rotating this region about the y-axis:

Solution:



This time let's make conical slices

(The revolution is cut into "shells")

$$V = \int 2\pi x h\,dx$$

$$V = \int_0^{\pi/4} 2\pi x \sin x\,dx + \int_{\pi/4}^{\pi/2} 2\pi x \cos x\,dx$$

How would you solve these integrals? IBP ??

YOU CONFIRM:

$$V = \pi^2\left(1 - \frac{1}{\sqrt{2}}\right)$$

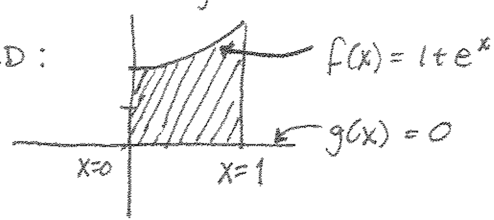
[Ex] (HW#22)

- Find the volume of the solid obtained by rotating this region about the y-axis:

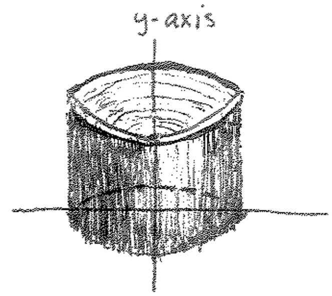
$y = 1 + e^x \qquad x = 0$

$y = 0 \qquad\quad\;\; x = 1$

In 2D:

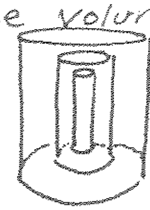


In 3D:

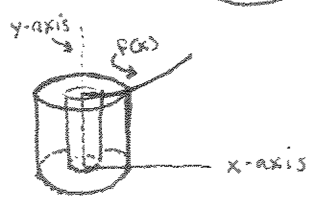


SOLUTION:

This time, let's sum up a series of conical slices in order to find the volume.



← These are the "shells" they are referring to in the book.

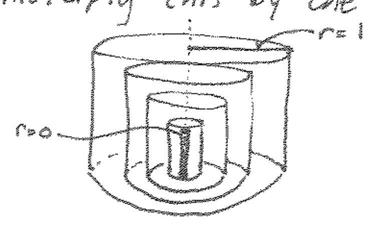


* Notice that the height of these shells is dictated by $y = 1 + e^x$.

- Let $h = 1 + e^x$
- We are going to integrate from $x = 0$ to $x = 1$.
  - THAT gives us this much:

- If we now multiply this by the circumference of a circle

$C = 2\pi r$



where the radius is dictated by the location of the "shell" on the x-axis, so $r = x$

$\Rightarrow C = 2\pi x$

(our shells are described by the equation $2\pi x h$.)

This leads us to the equation

$$V = \int_0^1 2\pi x h \, dx$$

for the whole volume.

and for our case, $V = \int_0^1 2\pi x(1+e^x)\,dx$.

$$\Rightarrow V = \pi x^2 + 2\pi(xe^x - e^x)\Big|_0^1$$

$$= \underline{3\pi}$$

$\boxed{EX}$ Find the length of one arch of the cycloid
$x = r(\theta - \sin\theta)$, $y = r(1-\cos\theta)$.

Solution: $\frac{dx}{d\theta} = r(1-\cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = r\sin\theta$

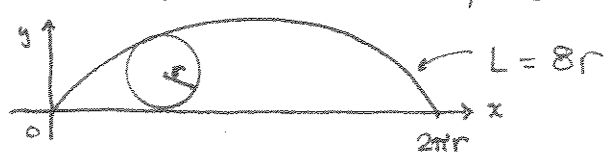
$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(r^2(1-2\cos\theta + \cos^2\theta + \sin^2\theta))}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

YOU show → $\vdots$ $= 8r$. (use your trig muscles)

This says that the length of one arch of the cycloid is 8 times the radius of the generating circle.



KEY IDEAS

- The length of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$s = \int ds = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx$$

- The length of a parametric curve is an integral w.r.t. t

$$s = \int ds = \int \left(\frac{ds}{dt}\right) dt = \int \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

CAN YOU EXPLAIN THE DIFFERENCE HERE?
(where does it come from?)

$\boxed{EX}$ What integral gives the length of the portion of the hyperbola $xy = 1$ from $(1,1)$ to $(2, 1/2)$? (It is impossible to evaluate this integral exactly, so just write down the integral...)