~ Predictable averages from random events ~

Ex

Say you throw a pair of dice. To begin, what are all the possible outcomes?

Answer: 6 sides, 2 dice, **2** through **12** are all the possible outcomes

Next question, what are the number of ways for each outcome?

Answer: We could count by brute force...

2 : ← 1 way
3 : or ← 2 ways ← ugh, takes a long time...
⋮

or, we could use this table:

1st die \ 2nd die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

← 2nd die; 1st die →; ← sums

THESE are all the possible Combinations. Count the # of ways by summing along the diagonal...
ex: 3 ways to get a 4
4 ways to get a 5...
etc...

so the probability of rolling a 6?
5/36.

the probability of rolling a 9?
4/36

OUTCOME:	2	3	4	5	6	7	8	9	10	11	12
PROBABILITY:	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

← NOTE: SHOULD SUM TO 1...

→ TICE E EAUTIFUL MMETRY!

The EXPECTED VALUE is found by multiplying the outcome & the probability of outcome and adding all...

$$\frac{2}{36}+\frac{6}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36}=7$$

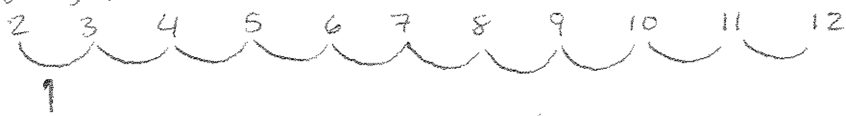
meaning that if you throw the dice 1000 times, the average would be between 6.9 and 7.1.

(if you did 100,000 trials, between 6.999 & 7.001....)

Now let's look at the "continuous" case...

The expected value $E(x) = \int_2^{12} x\, p(x)\, dx$

* if all numbers between 2 and 12 are equally probable...



the probability of an outcome between 2 and 3 is $1/10$
the probability of an outcome between x and $x+\Delta x$ is $\Delta x/10$
(Here, $\Delta x = 1$)

The probability of an outcome between x and $x + dx$ **is** $p(x)dx$

The problem above has $p(x) = 1/10$. ("Probability Density")

$$E(x) = \int_2^{12} x \tfrac{1}{10}\, dx = \left.\frac{x^2}{20}\right|_2^{12} = 7 = x_{avg.}$$

(Note: We are integrating over all possible outcomes!) //

OK. Let's kick it up a notch...

EX THREE DICE !!! (A homework problem)

Number of OUTCOMES? Range from $1+1+1=3$ to $6+6+6=18$.
3 – 18 ← all possible outcomes.

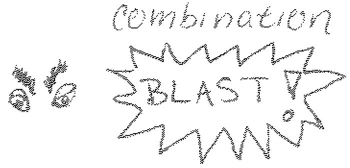
To count the number of ways for each outcome,
we might think about using Pascal's Triangle...

Ahh, Pascal. You so crazy.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
⋮

But wait!

Pascal's triangle breaks down, due to the constraint of our problem. Namely, a limited number of faces on the die makes it so that $(7+1+1)$ isn't a possible combination for us. (No 7 on the die...)



But wait...

Let's use SYMMETRY and some of Pascal's triangle and also our table from page 1! We will prevail!

Possible outcome:	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
# of ways:	1	3	6	10	15	21					21	15	10	6	3	1

(3–8: From Pascal; 9–12: USE table...; 14–18: From symmetry)

How many ways to roll a 9?

The third die:	1	2	3	4	5	6
in order to sum to 9, the other 2 dice must sum to	8	7	6	5	4	3
	= 9	= 9	= 9	= 9	= 9	= 9

From the table (counting along the diagonal)
there are 5 ways to roll an 8 with 2 dice.
" " 6 " " " a 7 " " "
" " 5 " " " " 6 " " "
" " 4 " " " " 5 " " "
" " 3 " " " " 4 " " "
" " 2 " " " " 3 " " "

So take $5 + 6 + 5 + 4 + 3 + 2 = 25$

$\therefore$ 25 ways to roll a 9 with 3 dice.

How many ways to roll a 10?

Do the same process: (1 2 3 4 5 6) ← 3rd die
sum of other 2 guys → (9 8 7 6 5 4)
4 5 6 5 4 3

$4 + 5 + 6 + 5 + 4 + 3$
$= 27$ ways to roll a 10.

- ok. Now use symmetry for 11 and 12.

So... 3 dice $\Rightarrow (\frac{1}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{1}{216}$

Possible outcome:	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Probabilities:	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{6}{216}$	$\frac{10}{216}$	$\frac{15}{216}$	$\frac{21}{216}$	$\frac{25}{216}$	$\frac{27}{216}$	$\frac{27}{216}$	$\frac{25}{216}$	$\frac{21}{216}$	$\frac{15}{216}$	$\frac{10}{216}$	$\frac{6}{216}$	$\frac{3}{216}$	$\frac{1}{216}$

To check: Make sure probabilities of all possible outcomes sum to 1.

Finding EXPECTED VALUE in same way as on page 1, (Method #1)
$= 10.5$

Again, we can find the expected value by integrating (way less painful, right?)

(Method # 2)

- we have 15 bins between x and x + Δx.
- our $\Delta x = 1$.

$$P(x) = 1/15 \longleftarrow \text{Probability density}$$

$$E(x) = \int_3^{18} x \cdot P(x)\,dx = \int_3^{18} x\left(\frac{1}{15}\right)dx = \left.\frac{x^2}{30}\right|_3^{18} = \frac{324}{30} - \frac{9}{30}$$

$$= 10.5$$ //