START HERE

3. $\int \frac{2x+4}{x^2-1}dx$ is the sum of the previous two integrals. Add A's and B's:

$$\frac{2x+4}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{-1}{x+1} + \frac{3}{x-1}.$$

In practice I would find $A = -1$ and $B = 3$ by the usual cover-up:

$$x = 1 \text{ gives } B = \frac{2x+4}{(x+1)} = \frac{6}{2}. \qquad x = -1 \text{ gives } A = \frac{2x+4}{(x-1)} = \frac{2}{-2}.$$

The integral is immediately $-\ln|x+1| + 3\ln|x-1|$. In this problem partial fractions is much better than substitutions. This case is $\frac{\text{linear}}{\text{quadratic}} = \frac{\text{degree 1}}{\text{degree 2}}$. *That is where partial fractions work best.*
The text solves the logistic equation by partial fractions. Here are more difficult ratios $\frac{P(x)}{Q(x)}$.

- It is the algebra, not the calculus, that can make $\frac{P(x)}{Q(x)}$ difficult. A reminder about division of polynomials may be helpful. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, *you first divide Q into P.* The example $\frac{x^3}{x^2+2x+1}$ requires long division:

$$\begin{array}{lrl} & x & \text{divide } x^2 \text{ into } x^3 \text{ to get } x \\ x^2+2x+1 & \sqrt{x^3} & \\ & x^3+2x^2+x & \text{multiply } x^2+2x+1 \text{ by } x \\ \hline & -2x^2-x & \text{subtract from } x^3 \end{array}$$

The first part of the division gives x. If we stop there, division leaves $\frac{x^3}{x^2+2x+1} = x + \frac{-2x^2-x}{x^2+2x+1}$. This new fraction is $\frac{\text{degree 2}}{\text{degree 2}}$. So the division has to continue one more step:

$$\begin{array}{lrl} & x-2 & \text{divide } x^2 \text{ into } -2x^2 \text{ to get } -2 \\ x^2+2x+1 & \sqrt{x^3} & \\ & x^3+2x^2+x & \\ \hline & -2x^2-x & \\ & -2x^2-4x-2 & \text{multiply } x^2+2x+1 \text{ by } -2 \\ \hline & 3x+2 & \text{subtract to find remainder} \end{array}$$

Now stop. The remainder $3x+2$ has lower degree than x^2+2x+1:

$$\frac{x^3}{x^2+2x+1} = x - 2 + \frac{3x+2}{x^2+2x+1} \text{ is ready for partial fractions.}$$

Factor x^2+2x+1 into $(x+1)^2$. Since $x+1$ is repeated, we look for

$$\frac{3x+2}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \quad \text{(notice this form!)}$$

Multiply through by $(x+1)^2$ to get $3x+2 = A(x+1)+B$. Set $x = -1$ to get $B = -1$. Set $x = 0$ to get $A + B = 2$. This makes $A = 3$. The algebra is done and we integrate:

$$\begin{aligned} \int \frac{x^3}{x^2+2x+1}dx &= \int \left(x - 2 + \frac{3x+2}{x^2+2x+1}\right)dx = \int \left(x - 2 + \frac{3}{x+1} - \frac{1}{(x+1)^2}\right)dx \\ &= \frac{1}{2}x^2 - 2x + 3\ln|x+1| + (x+1)^{-1} + C. \end{aligned}$$