4. $\int \frac{x^2+x}{x^2-4}dx$ also needs long division. The top and bottom have equal degree 2:

$$\begin{array}{rll} & 1 & \text{divide } x^2 \text{ into } x^2 \text{ to get } 1 \\ x^2+0x-4 & \sqrt{x^2+x} & \\ & \underline{x^2+0x-4} & \text{multiply } x^2-4 \text{ by } 1 \\ & x+4 & \text{subtract to find remainder } x+4 \end{array}$$

This says that $\frac{x^2+x}{x^2-4} = 1 + \frac{x+4}{x^2-4} = 1 + \frac{x+4}{(x-2)(x+2)}$. To decompose the remaining fraction, let

$$\frac{x+4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiply by $x-2$ so the problem is $\frac{x+4}{x+2} = A + \frac{B(x-2)}{x+2}$. Set $x=2$ to get $A = \frac{6}{4} = \frac{3}{2}$. Cover up $x+2$ to get (in the mind's eye) $\frac{x+4}{x-2} = \frac{A(x+2)}{(x-2)} + B$. Set $x=-2$ to get $B = -\frac{1}{2}$. All together we have

$$\int \frac{x^2+x}{x^2-4}dx = \int (1 + \frac{\frac{3}{2}}{x-2} - \frac{\frac{1}{2}}{x+2})dx = x + \frac{3}{2}\ln|x-2| - \frac{1}{2}\ln|x+2| + C.$$

5. $\int \frac{7x^2+14x+15}{(x^2+3)(x+7)}dx$ requires no division. Why not? *We have degree* 2 *over degree* 3. Also x^2+3 cannot be factored further, so there are just two partial fractions:

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{A}{x+7} + \frac{Bx+C}{x^2+3}. \qquad \text{Use } Bx+C \text{ over a quadratic, not just } B\,!$$

Cover up $x+7$ and set $x=-7$ to get $\frac{260}{52} = A$, or $A=5$. So far we have

$$\frac{7x^2+14x+15}{(x^2+3)(x+7)} = \frac{5}{x+7} + \frac{Bx+C}{x^2+3}.$$

We can set $x=0$ (because zero is easy) to get $\frac{15}{21} = \frac{5}{7} + \frac{C}{3}$, or $C=0$. Then set $x=-1$ to get $\frac{8}{24} = \frac{5}{6} + \frac{-B}{4}$. Thus $B=2$. Our integration problem is $\int (\frac{5}{x+7} + \frac{2x}{x^2+3})dx = 5\ln|x+7| + \ln(x^2+3) + C$.

6. (Problem 7.5.25) By substitution change $\int \frac{1+e^x}{1-e^x}$ to $\int \frac{P(u)}{Q(u)}du$. Then integrate.

- The ratio $\frac{1+e^x}{1-e^x}dx$ does not contain polynomials. Substitute $u=e^x$, $du = e^x\,dx$, and $dx = \frac{du}{u}$ to get $\frac{(1+u)du}{(1-u)u}$. A perfect set-up for partial fractions!

$$\frac{1+u}{u(1-u)} = \frac{A}{u} + \frac{B}{1-u} = \frac{1}{u} + \frac{2}{1-u}.$$

The integral is $\ln\ u - 2\ln|1-u| = x - 2\ln|1-e^x| + C$.