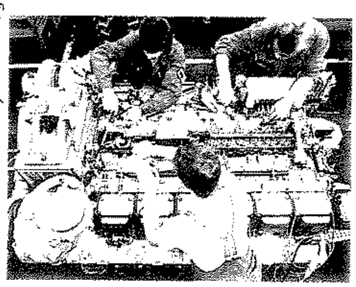
- How to recognize partial fraction decompositions of rational expressions
- How to find partial fraction decompositions of rational expressions

Partial fractions can help you analyze the behavior of a rational function. For instance, in Exercise 57 on page 185, you can analyze the exhaust temperatures of a diesel engine using partial fractions.



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Introduction

In this section, you will learn to write a rational expression as the sum of two or more simpler rational expressions. For example, the rational expression

$$\frac{x+7}{x^2-x-6}$$

can be written as the sum of two fractions with first-degree denominators. That is,

$$\underbrace{\frac{x+7}{x^2-x-6} = \overbrace{\underbrace{\frac{2}{x-3}}_{\text{Partial fraction}} + \underbrace{\frac{-1}{x+2}}_{\text{Partial fraction}}}^{\text{Partial fraction decomposition of } \frac{x+7}{x^2-x-6}}}_{}.$$

Each fraction on the right side of the equation is a **partial fraction,** and together they make up the **partial fraction decomposition** of the left side.

1. *Divide if improper:* If $N(x)/D(x)$ is an improper fraction [degree of $N(x) \geq$ degree of $D(x)$], divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{polynomial}) + \frac{N_1(x)}{D(x)}$$

and apply Steps 2, 3, and 4 below to the proper rational expression $N_1(x)/D(x)$. Note that $N_1(x)$ is the remainder from the division of $N(x)$ by $D(x)$.

2. *Factor the denominator:* Completely factor the denominator into factors of the form

$$(px+q)^m \quad \text{and} \quad (ax^2+bx+c)^n$$

where (ax^2+bx+c) is irreducible.

3. *Linear factors:* For *each* factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m}$$

4. *Quadratic factors:* For *each* factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

Section A.4, shows you how to combine expressions such as

$$\frac{1}{x-2} + \frac{-1}{x+3} = \frac{5}{(x-2)(x+3)}.$$

The method of partial fractions shows you how to reverse this process.

$$\frac{5}{(x-2)(x+3)} = \frac{?}{x-2} + \frac{?}{x+3}$$

Partial Fraction Decomposition

Algebraic techniques for determining the constants in the numerators of partial fractions are demonstrated in the examples that follow. Note that the techniques vary slightly, depending on the type of factors of the denominator: linear or quadratic, distinct or repeated.

Technology

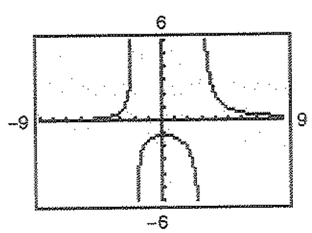
You can use a graphing utility to check *graphically* the decomposition found in Example 1. To do this, graph

$$y_1 = \frac{x+7}{x^2 - x - 6}$$

and

$$y_2 = \frac{2}{x-3} + \frac{-1}{x+2}$$

in the same viewing window. The graphs should be identical, as shown below.



Example 1 ▶ Distinct Linear Factors

Write the partial fraction decomposition of $\dfrac{x+7}{x^2 - x - 6}$.

Solution

The expression is not improper, so factor the denominator. Because $x^2 - x - 6 = (x-3)(x+2)$, you should include one partial fraction with a constant numerator for each linear factor of the denominator. Write the form of the decomposition as follows.

$$\frac{x+7}{x^2 - x - 6} = \frac{A}{x-3} + \frac{B}{x+2}$$ Write form of decomposition.

Multiplying each side of this equation by the least common denominator, $(x-3)(x+2)$, leads to the **basic equation**

$$x + 7 = A(x+2) + B(x-3).$$ Basic equation

Because this equation is true for all x, you can substitute any *convenient* values of x that will help determine the constants A and B. Values of x that are especially convenient are ones that make the factors $(x+2)$ and $(x-3)$ equal to zero. For instance, let $x = -2$. Then

$$-2 + 7 = A(-2 + 2) + B(-2 - 3)$$ Substitute -2 for x.

$$5 = A(0) + B(-5)$$

$$5 = -5B$$

$$-1 = B.$$

To solve for A, let $x = 3$ and obtain

$$3 + 7 = A(3 + 2) + B(3 - 3)$$ Substitute 3 for x.

$$10 = A(5) + B(0)$$

$$10 = 5A$$

$$2 = A.$$

So, the decomposition is

$$\frac{x+7}{x^2 - x - 6} = \frac{2}{x-3} + \frac{-1}{x+2}$$

as indicated at the beginning of this section. Check this result by combining the two partial fractions on the right side of the equation, or by using your graphing utility.

The next example shows how to find the partial fraction decomposition of rational expression whose denominator has a repeated linear factor.

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Write the partial fraction decomposition of $\dfrac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x}$.

This rational expression is improper, so you should begin by dividing the numerator by the denominator to obtain

$$x + \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}.$$

Because the denominator of the remainder factors as

$$x^3 + 2x^2 + x = x(x^2 + 2x + 1) = x(x + 1)^2$$

you should include one partial fraction with a constant numerator for each power of x and $(x + 1)$ and write the form of the decomposition as follows.

$$\frac{5x^2 + 20x + 6}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2}$$

Multiplying by the LCD, $x(x + 1)^2$, leads to the basic equation

$$5x^2 + 20x + 6 = A(x + 1)^2 + Bx(x + 1) + Cx.$$

Letting $x = -1$ eliminates the A- and B-terms and yields

$$5(\quad)^2 + 20(\quad) + 6 = A(\quad + 1)^2 + B(\quad)(\quad + 1) + C(\quad)$$

$$5 - 20 + 6 = 0 + 0 - C$$

$$C = 9.$$

Letting $x = 0$ eliminates the B- and C-terms and yields

$$5(\ )^2 + 20(\ ) + 6 = A(\ + 1)^2 + B(\ )(\ + 1) + C(\ )$$

$$6 = A(1) + 0 + 0$$

$$6 = A.$$

At this point, you have exhausted the most convenient choices for x, so to find the value of B, use *any other value* for x along with the known values of A and C. So using $x = 1$, $A = 6$, and $C = 9$,

$$5(\ )^2 + 20(\ ) + 6 = (\ + 1)^2 + B(\ )(\ + 1) + (\ )$$

$$31 = 6(4) + 2B + 9$$

$$-2 = 2B$$

$$-1 = B.$$

Therefore, the partial fraction decomposition is

$$\frac{x^4 + 2x^3 + 6x^2 + 20x + 6}{x^3 + 2x^2 + x} = x + \frac{6}{x} + \frac{-1}{x + 1} + \frac{9}{(x + 1)^2}.$$


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Historical Note
John Bernoulli (1667–1748), a Swiss mathematician, introduced the method of partial fractions and was instrumental in the early development of calculus. Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

The procedure used to solve for the constants in Examples 1 and 2 works well when the factors of the denominator are linear. However, when the denominator contains irreducible quadratic factors, you should use a different procedure, which involves writing the right side of the basic equation in polynomial form and *equating the coefficients* of like terms.

Example 3 ▶ **Distinct Linear and Quadratic Factors**

Write the partial fraction decomposition of

$$\frac{3x^2 + 4x + 4}{x^3 + 4x}.$$

Solution

This expression is not proper, so factor the denominator. Because the denominator factors as

$$x^3 + 4x = x(x^2 + 4)$$

you should include one partial fraction with a constant numerator and one partial fraction with a linear numerator and write the form of the decomposition as follows.

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by the LCD, $x(x^2 + 4)$, yields the basic equation

$$3x^2 + 4x + 4 = A(x^2 + 4) + (Bx + C)x. \qquad \text{Basic equation}$$

Expanding this basic equation and collecting like terms produces

$$\begin{aligned} 3x^2 + 4x + 4 &= Ax^2 + 4A + Bx^2 + Cx \\ &= (A + B)x^2 + Cx + 4A. \qquad \text{Polynomial form} \end{aligned}$$

Finally, because two polynomials are equal if and only if the coefficients of like terms are equal,

$$3x^2 + 4x + 4 = (A + B)x^2 + Cx + 4A \qquad \text{Equate coefficients of like terms.}$$

you obtain the equations

$$3 = A + B, \qquad 4 = C, \qquad \text{and} \qquad 4 = 4A.$$

So, $A = 1$ and $C = 4$. Moreover, substituting $A = 1$ in the equation $3 = A + B$ yields

$$\begin{aligned} 3 &= 1 + B \\ 2 &= B. \end{aligned}$$

Therefore, the partial fraction decomposition is

$$\frac{3x^2 + 4x + 4}{x^3 + 4x} = \frac{1}{x} + \frac{2x + 4}{x^2 + 4}.$$

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the *zeros* of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in step 1 above to rewrite the basic equation. Then substitute *other* convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like terms to obtain equations involving A, B, C, and so on.
4. Use substitution to solve for $A, B, C, \ldots$.

Keep in mind that for *improper* rational expressions such as

$$\frac{N(x)}{D(x)} = \frac{2x^3 + x^2 - 7x + 7}{x^2 + x - 2}$$

you must first divide before applying partial fraction decomposition.

Activities

1. Write the partial fraction decomposition of $\frac{5x - 10}{(x + 2)(2x - 1)}$.

   Answer: $\frac{4}{x + 2} - \frac{3}{2x - 1}$

2. Write the partial fraction decomposition of $\frac{4x^3 + 9x^2 - 2x + 6}{x(x + 2)}$.

   Answer: $4x + 1 + \frac{3}{x} - \frac{7}{x + 2}$

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Error Analysis Suppose you are tutoring a student in algebra. In trying to find a partial fraction decomposition, your student writes the following.

$$\frac{x^2 + 1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1}$$

$$\frac{x^2 + 1}{x(x - 1)} = \frac{A(x - 1)}{x(x - 1)} + \frac{Bx}{x(x - 1)}$$

$$x^2 + 1 = A(x - 1) + Bx \qquad \text{Basic equation}$$

By substituting $x = 0$ and $x = 1$ into the basic equation, your student concludes that $A = -1$ and $B = 2$. However, in checking this solution, your student obtains the following.

$$\frac{-1}{x} + \frac{2}{x - 1} = \frac{(-1)(x - 1) + 2(x)}{x(x - 1)}$$

$$= \frac{x + 1}{x(x - 1)}$$

$$\neq \frac{x^2 + 1}{x(x - 1)}$$

What has gone wrong?

The next example shows how to find the partial fraction decomposition of a rational function whose denominator has a *repeated* quadratic factor.

Additional Examples

Partial fraction decompositions can be confusing to students. You may want to go over several additional examples.

a. $\dfrac{2x^2 + 7x + 4}{(x + 1)^3} = \dfrac{2}{x + 1} + \dfrac{3}{(x + 1)^2} - \dfrac{1}{(x + 1)^3}$

b. $\dfrac{-x^3 + 4x^2 - 2x + 6}{x^2(x^2 + 2)} = -\dfrac{1}{x} + \dfrac{3}{x^2} + \dfrac{1}{x^2 + 2}$

c. $\dfrac{3x^3 - x^2 + 5x + 1}{(x^2 + 1)^2} = \dfrac{3x - 1}{x^2 + 1} + \dfrac{2x + 2}{(x^2 + 1)^2}$

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Write the partial fraction decomposition of

$$\frac{8x^3 + 13x}{(x^2 + 2)^2}.$$

You need to include one partial fraction with a linear numerator for each power of $(x^2 + 2)$.

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$$

Multiplying by the LCD, $(x^2 + 2)^2$, yields the basic equation

$$\begin{aligned} 8x^3 + 13x &= (Ax + B)(x^2 + 2) + Cx + D && \text{Basic equation} \\ &= Ax^3 + 2Ax + Bx^2 + 2B + Cx + D \\ &= Ax^3 + Bx^2 + (2A + C)x + (2B + D). && \text{Polynomial form} \end{aligned}$$

Equating coefficients of like terms

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A + C)x + (2B + D)$$

produces

$$8 = A,\ 0 = B,\ 13 = 2A + C, \text{ and } 0 = 2B + D. \qquad \text{Equate coefficients}$$

Finally, use the values $A = 8$ and $B = 0$ to obtain the following.

$$\begin{aligned} 13 &= 2A + C \\ &= 2(\ \ ) + C \\ -3 &= C \end{aligned}$$

$$\begin{aligned} 0 &= 2B + D \\ &= 2(\ \ ) + D \\ 0 &= D \end{aligned}$$

Check students' understanding. Why do you use two different procedures? When is each procedure more effective? Is it mathematically correct (although not necessarily as efficient) to use either procedure regardless of whether the factors of the denominator are linear or quadratic?

Therefore,

$$\frac{8x^3 + 13x}{(x^2 + 2)^2} = \frac{8x}{x^2 + 2} + \frac{-3x}{(x^2 + 2)^2}.$$

By equating coefficients of like terms in Examples 3 and 4, you obtained several equations involving A, B, C, and D, which were solved by *substitution*. In a later chapter you will study a more general method for solving such *systems of equations*.