~ Derivatives of $y = b^x$ and $x = \log_b y$ ~

Q: What is $\frac{dy}{dx}$ when $y = b^x$?

A: $\frac{dy}{dx} = \lim_{h \to 0} \left[\frac{b^{x+h} - b^x}{h} = \frac{b^x b^h - b^x}{h} = \frac{b^x(b^h - 1)}{h} \right]$

$$= b^x \lim_{h \to 0} \left[\frac{b^h - 1}{h} \right] = c\, b^x \;//$$

$dy/dx = c b^x = cy$. $\quad y' = cy \leftarrow$ A VERY IMPORTANT DE.

- It tells us that the rate of change is proportional to y.

Q: what is $\frac{dx}{dy}$ when $x = \log_b y$?

A: If $dy/dx = c b^x$, then, using chain rule...

$x = f(y) = f(b^x)$, CHAIN RULE $\quad f'(b^x)(c b^x) = 1$

$$\Rightarrow f'(b^x) = 1/cb^x = \frac{1}{cy} \;//$$

- The letter e denotes the unique positive real number such that $\ln(e) = 1$.

~ Growth & Decay ~

$$\frac{dy}{dt} = ky$$

- if $k > 0$, "law of natural growth"
- if $k < 0$, "law of natural decay"

- Because it is separable, $\int \frac{dy}{y} = \int k\,dt$

$$\ln|y| = kt + c$$

$$\Rightarrow y = e^{kt+c} = e^c e^{kt} = \underline{\underline{Ae^{kt}}}$$

what is A? $\quad y(0) = Ae^0 = A$.

KEY: The exponential law $y = y_0 e^{kt}$ solves the DE $y' = ky$ starting from y_0.

$\boxed{\text{EX}}$ (FROM NOTES) If a cell grows at 10% per day, find the time T it takes for the cell to double in size.

[ANS: 7 days]

$\boxed{\text{EX}}$ (FROM NOTES) Find the decay constant for carbon-14 if $y = \frac{1}{2} y_0$ in $T = 5568$ years.

[ANS: $c = \frac{\ln[1/2]}{5568}$]

EX] • To have \$50,000 for college tuition in 20 years, what gift y_0 should a grandparent make now? (Assume $c = 10\%$)

ANS: $y = y_0 e^{ct}$ $\Rightarrow$ $50{,}000 = y_0 e^{(.1 * 20)}$

$\Rightarrow y_0 = \$6767$ //

~ Solving the DE
$dy/dt = cy + s$ ~

• if we substitute $y = Ae^{ct} + B$, the solution will have the form (exponential + constant).

Then $y' = cAe^{ct}$

Plugging y & y' into the DE,

$$cAe^{ct} = cAe^{ct} + Bc + s$$

We can find $B = -s/c$ and $A = y_0 + \frac{s}{c}$
(ASK YOURSELF WHY...)

We are led to the solution $y = (y_0 + \frac{s}{c})e^{ct} - \frac{s}{c}$ //

~ NEWTON'S LAW OF COOLING ~

Say the temperature of a body is y, The surrounding temperature is y_∞. If y starts at y_0, we know $y \to y_\infty$ over time.

Newton says: The rate is proportional to the difference $(y - y_\infty)$

$$\frac{dy}{dt} = k(y - y_\infty)$$

[The bigger the difference, the faster heat flows]

EX] Your thanksgiving turkey is at 40°F when it goes in the oven at 350°F at 10:00 am. At noon, it is 110°. When will the Turkey be done? (195°)

ANS: $y_\infty = 350$, $y_0 = 40$. First, find k.

(we know $y = 110$ at $t = 2$ hours)

$(110 - 350) = -310e^{2k}$ $\Rightarrow$ $k = -.128$

Therefore,

$$y = -310e^{-.128t} + 350$$

The turkey is done at $y = 195$.

$t = 5.4$ hours. //

ASK YOURSELF: What feature of the constant k makes the turkey's temperature approach the value y_∞ as $t \to \infty$? (NOTE: IMPORTANT)

- If $dy/dt = u(y)c(t)$, such that the D.E. is SEPARABLE, we can use "separation of variables" on it.

EX] $$\frac{dy}{dt} = cy + s$$

soln: get all the "y stuff" on one side, and all the "t stuff" on the other.

$$\underbrace{\int \frac{dy}{(cy+s)}}_{\text{u-sub}} = \int dt$$

(WORK THROUGH...)

⋮

$$y = ae^{ct} - \frac{s}{c} = \left[y_0 + \frac{s}{c}\right]e^{ct} - \frac{s}{c} \quad //$$

EX] Solve $dy/dt = y^2$.

~ THE LOGISTIC EQUATION ~

$$\frac{dy}{dt} = cy$$

- For population growth, we may need to account for <u>COMPETITION</u>.

(The growth rate c may depend on y!)

For the <u>logistic</u> <u>equation</u>, $c(y) = c - by$

$$\frac{dy}{dt} = cy \underbrace{- by^2}$$

Nonlinear effect stems from <u>interaction</u>.

(This equation mathematically describes growth vs. competition)

EX] Solve the logistic equation via separation of variables.

(*ESSENTIAL: "Most applications of calculus come through differential equations". :))

~DIFFERENCE EQUATIONS~

Continuous : $\frac{dy}{dt} = cy$ Discrete: $y(t+1) = a\,y(t)$

Starting at y_0, $y(1) = a y_0$, $y(2) = a^2 y_0$, $y(3) = a^3 y_0$ ...

[continuous $y' = cy \rightsquigarrow y = e^{ct}$. discrete $y(t+1) = a y(t) \rightsquigarrow y = a^t y_0$]

KEY ⇒ if $|a| > 1$ we have growth. if $|a| < 1$ we have decay.

~HYPERBOLIC EQUATIONS~

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

- Recall the parametric equations $x = \cos t$, $y = \sin t$ describe the unit circle. In a similar fashion, $x = \cosh(t)$, $y = \sinh(t)$ describe the right branch of the unit **hyperbola**: $x^2 - y^2 = 1$.

EX] Find $\frac{d}{dx}[\cosh^2(3x-1)]$.

- If hyperbolic functions can be expressed in terms of e^x, then the inverses can be expressed in terms of lns...

EX] Find $\cosh^{-1}(y)$. [Hint: $2y = e^x + e^{-x}$]

EX] Find $\tanh^{-1}(y)$.

- what formulas for $\cos(x)$ and $\sin(x)$ correspond to e^x?

$$\cos(x) = \frac{1}{2}[e^{ix} + e^{-ix}] \quad \text{and} \quad \sin(x) = \frac{1}{2i}[e^{ix} - e^{-ix}]$$

Multiply $\sin(x)$ by i and add it to $\cos(x)$ to arrive at Euler's Equation. What happens when $x = \pi$?