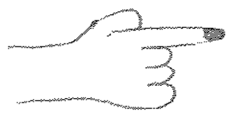


THE ROSETTA STONE:

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

This is how we go between
$(x, y) \longleftrightarrow (r, \theta)$

Here, we look at equations of the form $y = f(x)$.

Here, we look at equations of the form $r = F(\theta)$.

For Example,
$$(x, y) = (-1, \sqrt{3})$$
$$r = \sqrt{1+3} = 2, \quad \text{and} \quad \frac{y}{x} = \frac{\sqrt{3}}{-1} = \tan(\theta)$$

the function $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$ ← This is a fourth quadrant angle, while the point $(-1, \sqrt{3})$ is in the second quadrant!

$$-\pi/3 + \pi = \underline{2\pi/3}$$

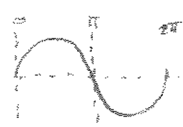
$$\therefore (r, \theta) = (2, 2\pi/3)$$

ASK YOURSELF: what other pairs of (r, θ) would give the same point?

Let's go in the other direction...

$\boxed{\text{EX}}$ if $(r, \theta) = (1, 3\pi/2)$, what is (x, y)?

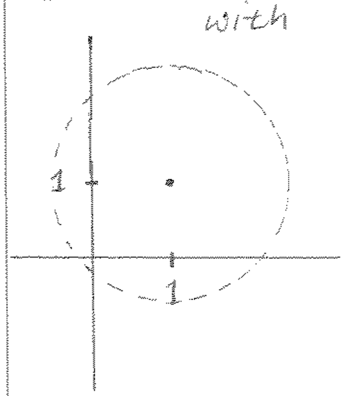
$x = 1\cos(\frac{3\pi}{2}) = 0$,



$y = 1\sin(\frac{3\pi}{2}) = -1$

$$(0, -1)$$

$\boxed{\text{EX}}$ Write the polar equation for a circle centered at $(1,1)$ with a radius of $\sqrt{2}$.



To begin, we know that the rectangular equation is $(x-1)^2 + (y-1)^2 = 2$

$$(x-1)(x-1) + (y-1)(y-1) = 2$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 = 2$$
$$x^2 - 2x + y^2 - 2y = 0$$

if we now replace x with $r\cos\theta$,

$$r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta - 2r\sin\theta = 0$$

$$r^2(\cancel{c}^2 + \cancel{s}^2) = r^2(1)$$

$$\Rightarrow r^2 = 2r\cos\theta + 2r\sin\theta$$

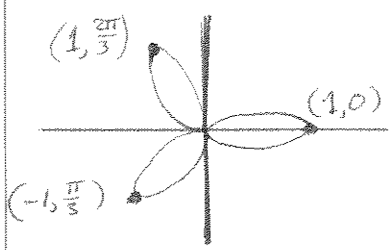
$$\therefore r = 2[\cos\theta + \sin\theta]$$

#2] Convert $r(\cos\theta - \sin\theta) = 2$ to xy coordinates...
ANS: This is the straight line, $x - y = 2$.

★ For finding symmetries,

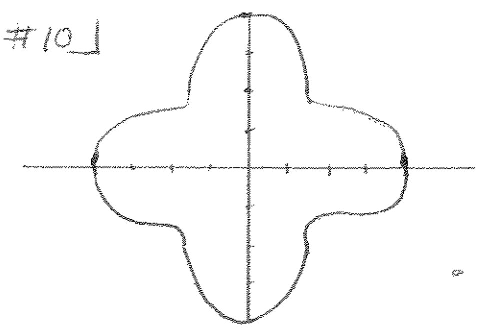
- x-axis symmetry: θ to $-\theta$ -OR- θ to $\pi - \theta$ and r to $-r$
- y-axis symmetry: θ to $\pi - \theta$ -OR- θ to $-\theta$ and r to $-r$
- polar symmetry: θ to $\pi + \theta$ -OR- r to $-r$

#9] Sketch the graph of $r = \cos(3\theta)$ and check for symmetry.



- Since $\cos(3\theta) = \cos(-3\theta)$, this curve is symmetric about the x-axis.
- Other symmetry tests fail.

#10]

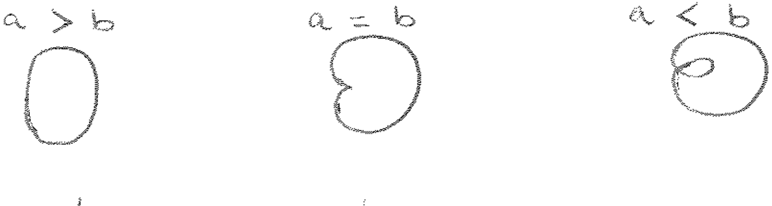


$r^2 = 10 + 6\cos(4\theta)$

- since $10 + 6\cos(4\theta) = 10 + 6\cos(-4\theta)$, we have x-axis symmetry ✓
- Since $10 + 6\cos(4\theta) = 10 + 6\cos(4[\pi - \theta])$ we have y-axis symmetry. ✓
- since $(-r)^2 = r^2$, we have polar symmetry. ✓

- For $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, the graphs of these equations are called "limaçons" (when $a = b$, they are "cartioids")

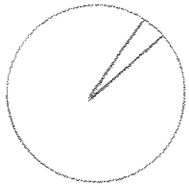
$a > b$ $\qquad$ $a = b$ $\qquad$ $a < b$



- For $r = \frac{d}{\cos\theta}$ or $r = \frac{d}{\sin\theta}$, we get lines
- For $r = 2a\cos(\theta)$ or $2a\sin(\theta)$, we get circles
- For $r = \frac{ed}{1 + e\cos(\theta)}$, we get ellipses, parabolas, hyperbolas...

ASK YOURSELF: How did we create petals?

- TRY $r = 1 - 2\sin(3\theta)$

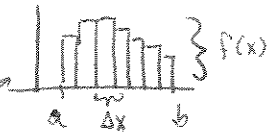


when $r = F(\theta)$, our "building blocks" will be wedges.

$\frac{\Delta\theta}{2\pi}$ = section, and Area $= \pi r^2$

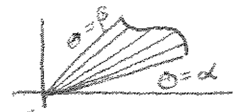
$\Rightarrow \Delta A$ = Area of section $= \frac{1}{2}[\Delta\theta] r^2$

so instead of summing up rectangles...



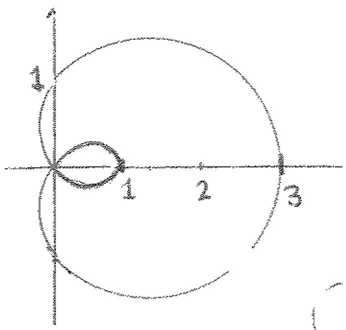
$\int_a^b f(x)dx$

we are summing up wedges...



$r = f(\theta) \Rightarrow \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 d\theta$

Ex] Find the area of $r = 1 + 2\cos\theta$ (inner loop only)



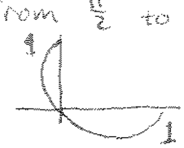
$A = \int \frac{1}{2} r^2 d\theta$

$= \int \frac{1}{2}[1 + 4\cos\theta + 4\cos^2\theta] d\theta$

But what are our limits of integration?

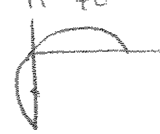
from 0 to $\frac{\pi}{2}$, we have

from $\frac{\pi}{2}$ to π, we have

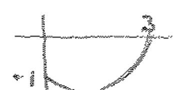


these are snapshots
★ Make sure you are comfortable plotting in Polar coordinates.

from π to $3\pi/2$,



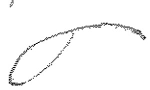
from $3\pi/2$ to 2π

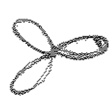


The inner loop is  found at $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$.

so this is what we will use for the limits of integration.
INTEGRATING, we get $\pi - \frac{3}{2}\sqrt{3}$.

Ex] $r = \cos(3\theta)$ (one petal only)

Notice that $0 \le \theta < \frac{\pi}{2}$ gives  and $\frac{\pi}{2} < \theta < \pi$ gives 

So 0 to π is  the full shape.

one petal is given when r begins and ends at $r = 0$, and goes around only once. (You would know this by constructing a table of values for θ & corresponding r, like in class...)
Thus $-\pi/6 < \theta < \pi/6$.

$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}\cos^2 3\theta \, d\theta = \frac{1}{4}\int [1 + 6\cos\theta] d\theta = \frac{\pi}{12}$$

§ 9.3 — ~TANGENTS~ — QUIZ # 4 REVIEW

- in cartesian, the slope of the tangent line $m = dy/dx$
- in polar,

$$\frac{dy}{dx} \Rightarrow \lim_{\Delta\theta \to 0} \frac{\Delta y/\Delta\theta}{\Delta x/\Delta\theta} = \frac{dy/d\theta}{dx/d\theta}$$

$$\frac{dy}{dx} = \frac{f(\theta)\cos(\theta) + f'(\theta)\sin(\theta)}{-f(\theta)\sin(\theta) + f'(\theta)\cos(\theta)}$$

(← can you answer WHY?)

~ Length of a polar curve ~

Recall $ds = \sqrt{(dx)^2 + (dy)^2}$

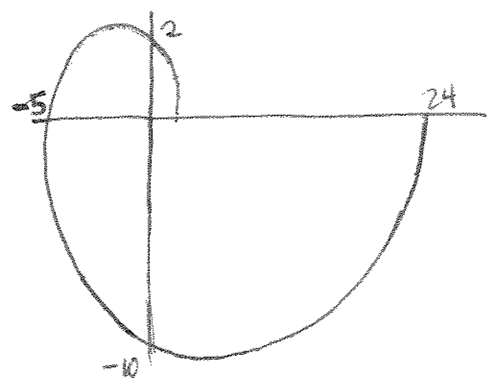
in polar, $r = f(\theta)$

$ds = \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$

HINT: can you get from here to here? (Look at your notes..)

EX] $r = e^{\theta/2}$

Find the length of the spiral as θ goes from 0 to 2π



soln:

$$\int \sqrt{\tfrac{1}{4}e^{\theta} + e^{\theta}}\, d\theta$$

$$= \int \sqrt{\tfrac{5}{4}e^{\theta}}\, d\theta = \int \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta = \sqrt{5}\, e^{\theta/2}\Big|_0^{2\pi} = \sqrt{5}(e^{\pi} - 1) \approx 49.5$$ //

Ex] $r = 1 + \cos\theta$

$$ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= \sqrt{(-\sin\theta)^2 + (1+\cos\theta)^2}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta} = \sqrt{2 + 2\cos\theta}$$

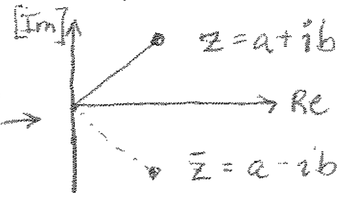
Total length is

$$\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta = 4\sin\frac{\theta}{2}\Big|_0^{2\pi} = 8$$

MAKE SURE THAT YOU KNOW HOW TO FIND YOUR LIMITS OF INTEGRATION. THIS MAY INVOLVE sketching the figure (by making the θ-r table) and seeing how much of $0 < \theta < 2\pi$ it takes to complete your figure...

- The important thing to note is that complex numbers aren't really all that "complex". The math works very much in the way that we would expect it to...

ARGAND DIAGRAM →



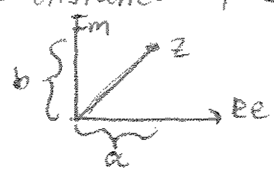
← complex conjugate

- If we have $\dfrac{a+ib}{c+id}$, multiply top & bottom by ...

COMPLEX NUMBER

$$\frac{a+ib}{c+id} \cdot \frac{(c-id)}{(c-id)} = \frac{ac - bd + i(bc - ad)}{c^2 + d^2} = \left[\frac{ac-bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]$$

- The distance of z from the origin...



$$|z| = \sqrt{a^2 + b^2} \quad \leftarrow \text{MODULUS}$$

(NOTE THAT $z\bar{z} = |z|^2$)

- if $(a,b) \to (r,\theta)$, using $a = r\cos\theta$, $b = r\sin\theta$

$$z = r(\cos\theta + i\sin\theta)$$

- Recall addition formulas

$$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$
$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$$

so that multiplying $z_1 \cdot z_2$ ...

$$z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

(multiply the r's and add the arguments.)

using this, we are led to DeMoivre's Theorem:

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

~ SOLUTIONS TO DIFFERENTIAL EQUATIONS ~

$$y'' = -4y$$

↓ solution is of the form $y = e^{ct}$

$$\Rightarrow c^2 = -4 \quad \Rightarrow \quad c = \pm 2i$$

Equation has 2 pure exponential solutions, $y_1 = e^{2it}$, $y_2 = e^{-2it}$.

- $y = Ae^{2it} + Be^{-2it}$ gives all solutions to the differential equation.

- The solution $y = e^{2it} = \cos 2t + i\sin 2t$ is complex. The DE is real. Take the real & imaginary parts of the complex solution

$y_{real} = \cos(2t)$ and $y_{im} = \sin(2t)$

These are "pure oscillatory solutions".

° Find all solutions of the form $y = e^{ct}$

#19] $y'' + y = 0$

ANS: $y'' = -y \Rightarrow c^2 = -1 \Rightarrow c = \pm i$ ← two c's here..

$y_1 = e^{it}$ is one solution. $y_2 = e^{-it}$ is another solution.

The combination $y = Ae^{it} + Be^{-it}$ gives the form of all solutions (we would choose A & B to satisfy our initial conditions, if we had any).

#20] $y''' + y = 0$

(LONG VERSION) ANS: $y''' = -y$, Now $c^3 = -1$ (There will be $K=3$ roots...)

$$\star \; -1 = e^{i\pi} \; \star$$

$$c^3 = -1 = e^{i\pi}$$

$$(c^3)^{1/3} = (e^{i\pi})^{1/3} = e^{i\pi/3}$$

$$\star\star \; e^{i\pi/3} = \left[\cos\left(\frac{\pi + 2\pi n}{3}\right) + i\sin\left(\frac{\pi+2\pi n}{3}\right)\right] \text{ for } n = 0,1,2 \; \star\star$$

We need 3 roots, THINK DEMOIVRE!

$n = 0 \Rightarrow e^{i\pi/3} = [\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})]$ ← c_1

$n = 1 \Rightarrow [\cos(3\pi/3) + i\sin(3\pi/3)] = e^{i\pi}$ ← c_2

$n = 2 \Rightarrow [\cos(5\pi/3) + i\sin(5\pi/3)] = e^{i5\pi/3}$ ← c_3

so $e^{i\pi/3}$ works, and so does $e^{i\pi}$ and $e^{i\frac{5\pi}{3}}$!

SIDE NOTE:

~ Finding roots of a complex number ~

Let $z = r(\cos\theta + i\sin\theta)$ and let K be a positive integer. Then z has the K distinct K^{th} roots

$$c_K = r^{1/K}\left[\cos\left[\frac{\theta + 2\pi n}{K}\right] + i\sin\left[\frac{\theta+2\pi n}{K}\right]\right]$$

where $n = 0, 1, 2 \ldots K-1$. //

so $c_1 = (\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$c_2 = (\cos(\pi) + i\sin(\pi)) = -1 + i \cdot 0 = -1$

$c_3 = (\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})) = \frac{1}{2} + i(-\frac{\sqrt{3}}{2}) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

The root $c_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ gives $y_1 = e^{(1+\sqrt{3}i)/2}$

The root $c_2 = e^{i\pi} = -1$ gives $y_2 = e^{-t}$

The root $c_3 = e^{i5\pi/3} = (1 - i\sqrt{3})/2$ gives $y_3 = e^{(1-\sqrt{3}i)/2}$

so all solutions to the DE are given by the combination

$$y = Ae^{(1+\sqrt{3}i)t/2} + Be^{-t} + Ce^{(1-\sqrt{3}i)t/2} \; //$$

*TIP: (For last page...)

Every complex number is

$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$

#21] $y''' - y' = 0$

ANS:

$c^3e^{ct} - ce^{ct} = 0 \Rightarrow e^{ct}(c^3 - c) = 0$

Factor $(c^3 - c) = 0 \Rightarrow c(c^2 - 1) = 0$

• one c will be zero. The other two are found by

$c^2 = 1 \Rightarrow c = \pm 1$

$c_1 = 0, \quad c_2 = 1, \quad c_3 = -1$

$\Rightarrow y_1 = e^0, \quad y_2 = e^t, \quad y_3 = e^{-t}$

so

$y = \underbrace{Ae^0}_{A} + Be^t + Ce^{-t}$ // (This one was easy!)

#22] $y'' + 6y' + 5y = 0$

ANS: if $y = e^{ct}$, $y' = ce^{ct}$, $y'' = c^2e^{ct}$

plugging in...

$[c^2e^{ct}] + 6[ce^{ct}] + 5[e^{ct}] = 0$

$\Rightarrow e^{ct}[c^2 + 6c + 5] = 0$. Solve for c.

$(c+5)(c+1) = 0$

$c_1 = -5$ and $c_2 = -1 \Rightarrow y = Ae^{-5t} + Be^{-t}$ //

#23] Construct 2 real solutions from the real & imaginary parts of e^{ct} (first find c).

$y'' + 49y = 0$

$\Rightarrow y'' = -49y$, meaning $c^2 = -49$, or $c = \pm 7i$

$e^{7it} = \cos(7t) + i\sin(7t)$

• Take real & imaginary parts of the complex solution, giving $\cos(7t)$ and $\sin(7t)$. //

— CHECK: Do these satisfy the DE? —

$(\cos(7t))'' = -49\cos(7t)$

$-49\cos(7t) + 49\cos(7t) = 0$ ✓

$(\sin(7t))'' = -49\sin(7t)$

$-49\sin(7t) + 49\sin(7t) = 0$ ✓

#24] $y'' - 2y' + 2y = 0$

ANS: if $y = e^{ct} \Rightarrow c^2e^{ct} - 2ce^{ct} + 2e^{ct} = 0$

$e^{ct}\underbrace{[c^2 - 2c + 2]}_{c_{1,2} = 1 \pm i} = 0$

$c_{1,2} = 1 \pm i \rightarrow$ Because $c_{1,2} = \frac{2 \pm \sqrt{4 - 4\cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \sqrt{-1} = 1 \pm i$ //

so $e^{ct} = e^{(1+i)t} = e^te^{it}$

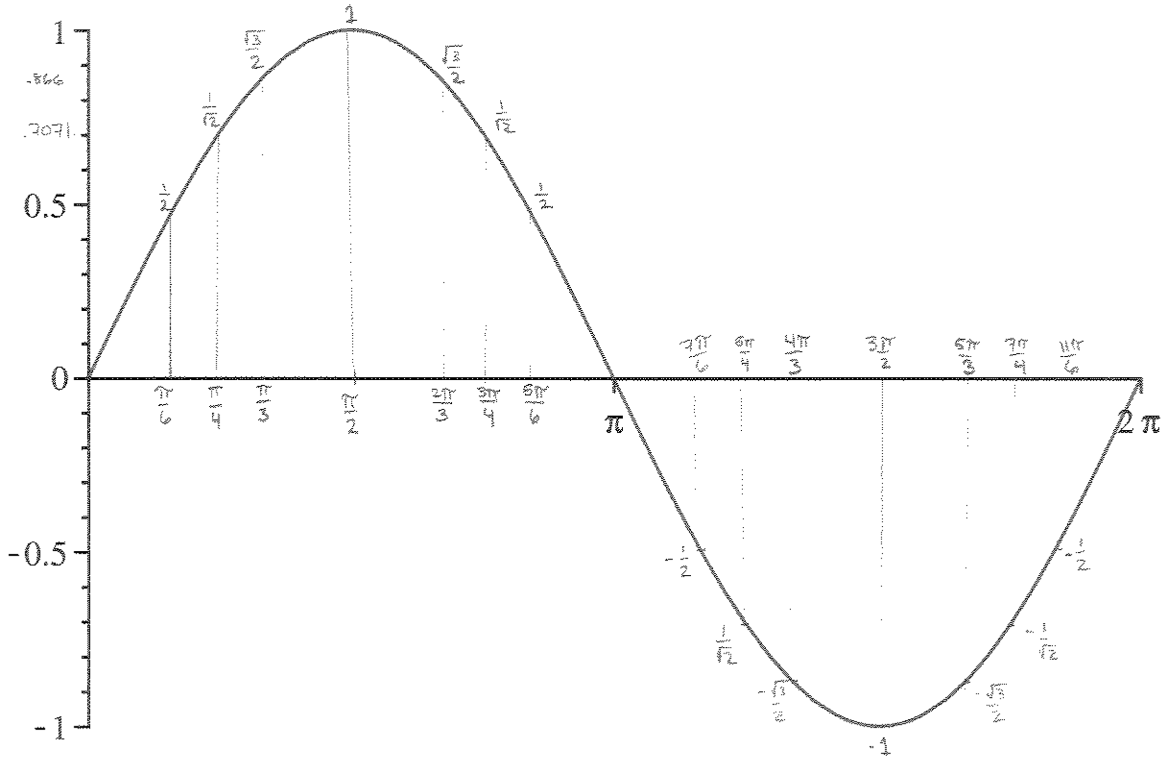
$e^te^{it} = e^t[\cos(t) + i\sin(t)]$

$= \underbrace{e^t\cos(t)} + i\,\underbrace{e^t\sin(t)}$

Two real solutions are therefore...

$e^t\cos(t)$ and $e^t\sin(t)$. //

```
> plot(sin(x), x = 0 .. 2*Pi);
```




```
> plot(cos(x), x = 0 .. 2*Pi);
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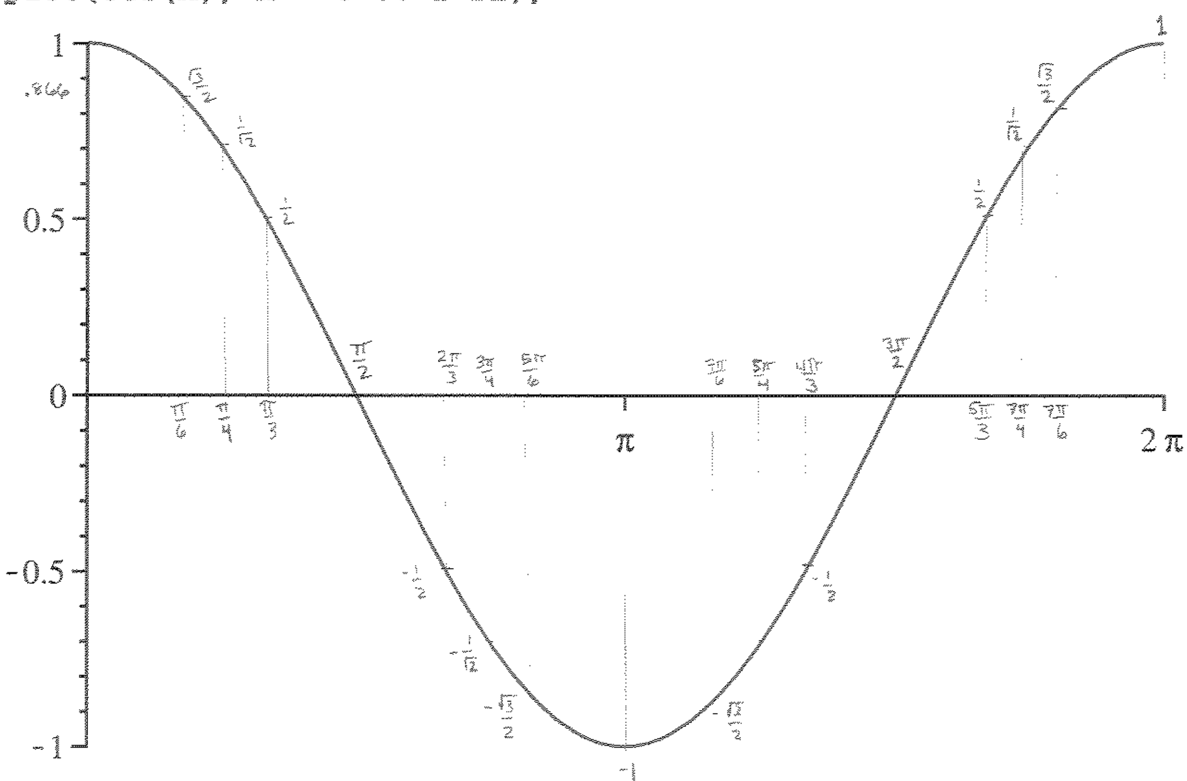


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