Solution to Example 6 from MATH 5010 Notes

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Abstract

This is a solution to Example 6 from the MATH 5010 notes.

1 Problem

Dave's Donuts offers 14 flavors of donuts (consider the supply of each flavor as being unlimited). The "grab bag" box consists of flavors randomly selected to be in the box, each flavor equally likely for each one of the dozen donuts. What is the probability that at most three flavors are in the grab bag box of a dozen?

2 Solution

For this we will need the multinomial distribution, which is a discrete probability distribution. In a string of characters there are k characters possible to fill one position of the string, which is n characters long. The random variable X_1 counts the number of occurences of character 1 in the string, X_2 the number of occurences of character 2, and so on until X_k. Let $p_1, \ldots, p_k$ be the individual probability each of the characters could appear in a position of the string; each position is filled independently of the characters in other positions. Let $x_1, \ldots, x_k \in [n] \cup \{0\}$ such that $x_1 + \ldots + x_k = n$. Then

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \ldots x_k!} p_1^{x_1} \ldots p_k^{x_k}.$$

Here, $k = 14$, $p_1 = \ldots = p_{14} = \frac{1}{14}$, and $n = 12$. So we can say

$$\mathbb{P}(X_1 = x_1, \ldots, X_{14} = x_{14}) = \frac{12!}{x_1! \ldots x_{14}!} \left(\frac{1}{14}\right)^{12}. \tag{1}$$

We will say

$$\begin{aligned}\mathbb{P}(\text{At most three flavors}) = &\mathbb{P}(\text{Exactly one flavor}) \\ &+ \mathbb{P}(\text{Exactly two flavors}) \\ &+ \mathbb{P}(\text{Exactly three flavors}).\end{aligned}$$

Compute each of those probabilities separately.

If $X_1 = 12$ and $X_2 = \ldots = X_{14} = 0$, there is exactly one flavor in the box. (1) shows the probability this happens is $\frac{1}{14^{12}}$. Since we could pick an $X_i = 12$ and there were 14 ways to make this decision, we can say

$$\mathbb{P}(\text{Exactly one flavor}) = \frac{1}{14^{11}}. \tag{2}$$

Let's now compute $\mathbb{P}(\text{Exactly two flavors})$. We start by fixing $X_3 = \ldots = X_{14} = 0$. We get

$$\mathbb{P}(X_3 = \ldots = X_{14} = 0) = \sum_{i=0}^{12} \binom{n}{i} \left(\frac{1}{14}\right)^{12} = \left(\frac{2}{14}\right)^{12}. \tag{3}$$

Unfortunately (3) includes cases where there's actually only one flavor present in the box, so compute

$$\begin{aligned}\mathbb{P}(X_1 \geq 1, X_2 \geq 1, X_3 = 0, \ldots, X_{14} = 0) &= \sum_{i=1}^{11} \binom{n}{i} \left(\frac{1}{14}\right)^{12} &(4)\\ &= \left(\frac{2}{14}\right)^{12} - 2\left(\frac{1}{14}\right)^{12}. &(5)\end{aligned}$$

Of course we could have picked different variables to fix at zero, and there were $\binom{14}{2}$ ways to pick the variables to fix at zero (or equivalently, pick the variables to not fix at zero), finally yielding

$$\mathbb{P}(\text{Exactly two flavors}) = \binom{14}{2}\left(\left(\frac{2}{14}\right)^{12} - 2\left(\frac{1}{14}\right)^{12}\right). \tag{6}$$

Now to compute $\mathbb{P}(\text{Exactly three flavors})$. Again we start by fixing $X_4 = \ldots = X_{14} = 0$ and compute

$$\mathbb{P}(X_1 \geq 1, X_2 \geq 1, X_3 \geq 1, X_4 = \ldots = X_{14} = 0) = \sum_{i=1}^{10}\sum_{j=1}^{11-i}\left(\frac{12!}{i!j!(12-i-j)!}\right)\left(\frac{1}{14}\right)^{12}. \tag{7}$$

We could try and use tricks to compute (7) or we can acknowledge that we're busy people and ask SymPy to do it. Check that the following Python code is correct:

```
from sympy import init_session, binomial
init_session()

def multinomial(params):
    if len(params) == 1:
        return 1
    return binomial(sum(params), params[-1]) * \
                    multinomial(params[:-1])

l1 = list()
for i in range(1, 10 + 1):
    v = sum([multinomial([i, j, (12 - i - j)]) for j in range(1,
                                                      11 - i + 1)])
    l1.append(v)

sum(l1)/14**12     # Solution
```

The resulting probability is $\frac{129789}{14173478093824}$. We could have picked different flavors to fix, and there were $\binom{14}{3}$ ways to pick the flavors to fix, so we get

$$\mathbb{P}(\text{Exactly three flavors}) = \binom{14}{3}\left(\frac{129789}{14173478093824}\right) = \frac{1687257}{506195646208}. \tag{8}$$

We can write (2) and (6) as $\frac{1}{4049565169664}$ and $\frac{26611}{4049565169664}$, respectively. Summing these probabilities yields

$$\mathbb{P}\left(\text{At most three flavors}\right) = \frac{3381167}{1012391292416} \approx 3.34 \times 10^{-6}. \tag{9}$$

3 Related Question

This is the proper way to obtain the probability that there are at most three flavors in the "grab bag" box, but how many boxes exist in which there are at most three flavors when we discount the number of ways there are to arrange the donuts in a box?

If there's exactly one flavor, then we pick it and fill the box with that flavor; there's 14 ways to pick one flavor. If there's exactly two flavors in the box, we'll call them Flavor 1 and Flavor 2. There is at least one donut of Flavor 1 and one of Flavor 2. Now pick the rest of the donuts' flavors, order doesn't matter, there is replacement; there are $\binom{10+2-1}{10} = \binom{11}{10} = 11$ ways to do that. Then pick the two flavors: there's $\binom{14}{2}$ ways to do that, and thus $11\binom{14}{2}$ boxes with exactly two flavors. Similarly, for exactly three flavors, there are $\binom{14}{3}\binom{9+3-1}{9} = \binom{14}{3}\binom{11}{9}$ ways for there to be exactly three flavors. Sum these numbers. (See `https://math.stackexchange.com/q/3230011`.) There are 21,035 such boxes.