Localized phase transition wave in a bistable lattice

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Abstract

Discrete square-cell and triangular-cell lattices with bistable bonds are considered. Initially, the bond follows the Hooke's law as the first branch of the force—elongation diagram, but when the elongation reaches the critical value, the tensile force drops and the transition to the other branch occurs. The latter branch can be placed in parallel with the former (mathematically this case is simpler) or have any inclination. In particular, it may correspond to a crack with free faces. In such an initially stresses structure, a spontaneous transition wave can propagate between neighboring lines of particles. Steady-state transition waves in infinite irreversible elastic and viscoelastic lattices are studied analytically, while numerical simulations are used for transient waves in finite reversible and irreversible lattices. Besides, the related continuous model is examined. The goal is to determine the conditions which allow such a wave to exist, the transition wave speed and the energy dissipation in such processes — in dependence of the structure parameters and the load type and level.

1 Introduction

Lattices consisting of particles connected by massless bistable bonds are considered. The bonds are assumed to obey a two-phase stress-strain relation, such that the phase transition is accompanied by a jump discontinuity in the tensile force. In addition, the phases may differ by the tangent modulus.

In such an initially stressed lattice, as a result of a disturbance, the phase transition can arise as a *localized wave* propagating between two neighboring lines of the lattice particles. This wave results in the unloading of the outer bonds which thus remain to be in the initial phase. So, we face a one-dimensional phase transition wave propagating in a two-dimensional lattice. It looks like a propagating crack bridged by fibres as in fracture of fibre-reinforced composites [Budiansky et al. (1986), Willis (1993), Meda and Steif (1994a,b), Movchan and Willis (1996, 1997a,b, 1998), Huang et al. (1999), Nemat-Nasser and Luqun (2001), Biner (2002)].

The present formulation integrate those used in the works on one-dimensional phase transition waves in bistable chains [Slepyan and Troyankina (1984, 1988), Puglisi and Truskinovsky (2000), Slepyan (2000, 2001b), Balk et al. (2001a,b), Charlotte and Truskinovsky (2002), Ngan and Truskinovsky (2002)] and fracture of two-dimensional lattices [Slepyan (1981, 2000, 2001a,c), Kulakhmetova et al. (1984), Fineberg et al. (1991, 1992), Marder and Liu (1993), Marder and Gross (1995), Kessler and Levine (1998, 2001), Fineberg and Marder (1999), Slepyan et al. (1999), Kessler (2000), Gerde and Marder (2001), Pechenik at al. (2002), Slepyan and Ayzenberg-Stepanenko (2002)]. The goal of this study is to determine conditions of the existence of such a transition wave, the dependence of its speed on the initial stress and the energy dissipated during the dynamic transition.

In a special case, where the phases differ only by a jump in the tensile force, while the tangent modulus is the same for both phases, the bonds are assumed to obey a standard-material viscoelastic law. In a general case, elastic bistable lattices are considered.

Square- and triangular-cell lattices are examined. Note that the square-cell lattice is intended to model anti-plane shear; however, a hypothetic plane deformation with no displacements in the x-direction — with the same formulation and results — can also be assumed. We use the latter viewpoint when it is more convenient; for example, when displacements and forces are to be shown on a plane figure.

2 Formulation

Consider a lattice which massless bonds obey the following force-elongation relation (Figure 1):

$$\begin{aligned} &T = \mu\, Q \quad (t < t_*), \quad T = \mu\, Q - P_* - (1-\gamma)\mu\,(Q - Q_c) = \gamma\mu\, Q - P^* \quad (t > t_*), \\ &P^* = P_* - (1-\gamma)\mu\, Q_c, \quad Q < Q_c \quad (t < t_*), \quad Q = Q_c \quad (t = t_*). \end{aligned} \tag{1}$$

In these relations, T is the tensile force in a bond, Q is the bond elongation and Q_c is the critical elongation. For $t < t_*$ the bond is in the initial phase characterized by stiffness μ. The phase transition occurs at $t = t_*$ when the elongation first reaches the critical value. At this moment a jump in the tensile force arises, $[T] = T(t_* + 0) - T(t_* - 0) = -P_* < 0$, and the bond comes into the other phase where the tangent 'modulus' $dT/dQ = \gamma\mu, \gamma > 0$. Note that the phase transition moment, $t - t_*$, is different for different bonds. A one-dimensional elastic chain with such an irreversible behavior of its bond was considered by Slepyan and Troyankina (1984, 1988). A reversible two-phase chain was considered by Balk et al. (2001a,b). The above relation and the below considerations concern to a lattice under extension. Clearly, the same formulation and results are applicable for the lattice under compression or shear.

Initially, the lattice (presented in Figure 2) is uniformly stressed in such a way that the elongation of the y-directed or inclined bonds (as in a square-cell or a triangular-cell lattice, respectively), Q_0, is below the critical value, Q_c, while the x-directed bonds are not stressed. Note that in the case of mode II state of a triangular-cell lattice, lengthening and shortening alternate. In this case, the critical value is assumed to be $\pm Q_c$ for these deformations, respectively.

It is reasonable to assume that the phase transition wave arisen by a disturbance is localized as a bridged crack between the neighboring lines of particles (denoted as $n = 0$ and $n = -1$). Indeed, the negative jump in the tensile force acts to unload the y-directed or inclined bonds outside this damaged layer, while the initially unstressed x-directed bonds have a great safety factor. It is also assumed that the wave propagates with a constant speed v. This means that the time-interval between the transformation of the neighboring bonds is a/v, where a is the lattice cell size. The steady-state regime is considered with the transformed layer at $\eta \equiv (x - vt)/a < 0$, where the particle coordinates are $x = am, m = 0, \pm 1, \dots$.

Thus, we face a steady-state problem for a lattice which bonds are in the initial phase except those placed between lines $n = 0$ and $n = -1$ at $\eta < 0$. A solution is required which satisfies the dynamic equations for such a 'damaged' lattice and the following inhomogeneous conditions at infinity: the strain field coincides with a given initial one at $y = \pm\infty$ and $\eta = +\infty$. The solution presents the elongation of a bond at the moment of the above-mentioned transformation, $\eta = 0$, as a function of the wave speed, v, and the initial elongation. The equality

$$Q(\eta) = Q_c \quad (\eta = 0) \tag{2}$$

serves for the determination of the speed as a function of the initial elongation.

For our goal it is convenient to reformulate this problem as follows. Consider a uniform lattice all the bonds of which are in the initial phase. Introduce self-equilibrated couples of external forces applied to the particles connected by the crack line bonds. Each couple of forces is directed along the corresponding bond outwards as shown in Figure 3. Referring to (1) choose the forces as

$$P(\eta) = P_* + (1 - \gamma)\mu\,([Q(\eta) - Q_c] \quad (\eta < 0), \quad P(\eta) = 0 \quad (\eta \geq 0). \tag{3}$$

The crack line intact bonds together with these external forces act on the 'crack-face' particles, $n = 0$ and $n = -1$, the same as if the bonds are in the second phase. Hence, such a reformulation does not concern the lattice dynamic behavior.

Note that the case

$$P_* = \mu\, Q_c, \quad Q_0 = \gamma = 0 \tag{4}$$

corresponds to a free-face crack considered in Slepyan (2001a,c). Another remarkable case, $\gamma = 1$, will be considered below.

3 Governing equation

To proceed introduce a steady-state fundamental solution, $\mathcal{Q}(\eta)$, corresponding to the above-mentioned self-equilibrated load. In these terms, the additional elongation of a crack-line bond caused by the crack-face external forces is

$$\Theta(\eta) = Q(\eta) - Q_0 = \mathcal{Q}(\eta) * P(\eta), \tag{5}$$

where $*$ is the convolution symbol. In terms of the Fourier transform, this equality becomes

$$\Theta^F(k) = \mathcal{Q}^F(k)P^F(k). \tag{6}$$

Denote

$$\Theta^F = \Theta_+ + \Theta_-, \quad \Theta_\pm = \int_{-\infty}^{\infty} \Theta(\eta)H(\pm\eta)e^{ik\eta}d\eta, \tag{7}$$

where H is the unit step function. The same subscript, + (−), will be used for the right-side (left-side) Fourier transform of any function. Referring to (3)

$$P^F = P_- = P_*(0+ik)^{-1} + (1-\gamma)\mu\left[\Theta_- - \Theta_c(0+ik)^{-1}\right], \tag{8}$$

where $\Theta_c = Q_c - Q_0$. Now equation (6) can be presented in the form

$$\Theta_+ + \left[1-(1-\gamma)\mu\,\mathcal{Q}^F\right]\Theta_- = [P_* - (1-\gamma)\mu\,\Theta_c]\,\frac{\mathcal{Q}^F}{0+ik}. \tag{9}$$

The fundamental solution, $\mathcal{Q}^F(k)$, appearing in this equation can be expressed in terms of the known lattice-with-a-crack Green functions denoted by $L(k)$ in Slepyan (2001a,c). With this in mind we rearrange relation (9) to obtain the corresponding equation for the lattice with a crack at $\eta < 0$. Represent

$$P^F(k) = \mu\,Q_-(k) + p^F(k), \tag{10}$$

where $p\,(\eta)$ is an arbitrary function. The first term in the right-hand side of (10) compensates the tensile forces in the considered bonds at $\eta < 0$ and hence, as far as the external forces are defined by Eq. (10), this corresponds to a lattice with a crack at $\eta < 0$ loaded at $n = 0$ and $n = -1$ by forces $p\,(\eta)$.

In the considered case $\Theta = Q$, since $Q_0 = 0$ (4), and Eq. (6) leads to

$$L(k)Q_+(k) + Q_-(k) = [L(k) - 1]\frac{p^F}{\mu} \tag{11}$$

with

$$L(k) = \left[1 - \mu\,\mathcal{Q}^F\right]^{-1}. \tag{12}$$

In terms of the elongations, Eq. (11) coincides with the governing equations for crack dynamics [see Eq. (36) in Slepyan (2001a) and Eq. (11) in Slepyan (2001c)]. Thus, the expressions for $L(k)$ presented in these papers [Eq. (37) and Eq. (53), respectively] can be used to find $\mathcal{Q}^F$ from (12). It is

$$\mathcal{Q}^F = \frac{1}{\mu}\left[1 - \frac{1}{L(k)}\right], \tag{13}$$

where for mode III unbounded square-cell lattice

$$L(k) = \frac{r(k)}{h(k)} \tag{14}$$

with

$$h(k) = \sqrt{2(1-\cos k)+Y}, \quad r(k) = \sqrt{h^2(k)+4} \tag{15}$$

and

$$Y = (0+ikV)^2, \quad 0+ikV = \lim_{s\to+0}(s+ikV), \quad V = \frac{v}{c}, \quad c = \sqrt{\frac{a^2\mu}{M}}. \tag{16}$$

Here and below a is the bond length and M is the lattice particle mass. Function $L(k)$ for a square-cell lattice strip can be found in Slepyan et al. (1999).

For a triangular-cell lattice

$$L(k) = \frac{h_\Delta(k)}{r_\Delta(k)}, \tag{17}$$

where

$$\begin{aligned} h_\Delta &= \frac{F(n_2)\sqrt{n_1^2-1} - F(n_1)\sqrt{n_2^2-1}}{n_2-n_1}, \\ r_\Delta &= 3\sqrt{n_1^2-1}\sqrt{n_2^2-1}, \\ F(n_{1,2}) &= 3(\cos k/2 - n_{1,2})^2 + 6\sin^2 k/2(1\pm\cos k/2)(1\mp n_{1,2}) \\ &\quad +Y[(1\pm\cos k/2)(1\mp n_{1,2}) + 1 - n_{1,2}\cos k/2]. \end{aligned} \tag{18}$$

In the expression for $F(n_{1,2})$, upper signs correspond to mode I, while lower signs correspond to mode II. Besides,

$$n_{1,2} = \left(1+2\sin^2 k/2 + \frac{2}{3}Y\right)\cos k/2 \mp \left[\frac{1}{9}Y^2 - 4\sin^2\frac{k}{2}\left(\sin^2\frac{k}{2}+\frac{1}{3}Y\right)^2\right]^{1/2}. \tag{19}$$

The rule of the determination of signs of the radicals in the above relations [the argument of a radical is continuous in a pre-limiting case, $s > 0$, see (16)], as well as the asymptotes for $k \to 0$ and $k \to \pm\infty$, can be obtained from the corresponding papers. In particular,

$$\frac{h(k)}{r(k)} \to \frac{h_\Delta(k)}{r_\Delta(k)} \to 1 \quad (k\to\pm\infty). \tag{20}$$

Eq. (9) can now be rewritten as

$$\mathcal{L}(k)\Theta_+ + \Theta_- = \frac{P^*[\mathcal{L}(k)-1]}{(1-\gamma)\mu\,(0+ik)} \tag{21}$$

with

$$P^* = P_* - (1-\gamma)\mu\,\Theta_c \tag{22}$$

and

$$\mathcal{L}(k) = \frac{L(k)}{\gamma L(k)+1-\gamma}. \tag{23}$$

Note that the initial strain does not appear in the governing equation (21). The problem may thus be considered as a homogeneous one, that is in terms of the additional elongations vanishing at $x \to +\infty$ and $y \to \pm\infty$, as if the force-elongation diagram has its origin at $Q = Q_0, T = \mu\, Q_0$ as shown in Figure 1 by dotted axes. In these terms, the force P in (3) is a function of Θ as the difference between the initial and the other phases, while P^* (22) is that at $\Theta = 0$ (Figure 4). An *effective initial elongation* can be introduced for $\gamma < 1$ as

$$Q^0 = \frac{P^*}{(1-\gamma)\mu} \tag{24}$$

which is also shown in Figure 4.

4 Solution for $0 < \gamma < 1$

Eq. (21) can be solved using the Wiener-Hopf technique. In this connection first note some features of the function $L(k)$ (23). Under the causality principle this function should be considered as the limit $(s \to +0)$ of $L(s + ikV, k)$ [see Slepyan (2001a)]. It follows from the *Theorem of the fundamental solution* [Slepyan (2001c)] that if $s > 0$, as k runs over the real axis from $-\infty$ to ∞, the function $L(k)$ forms a closed contour on the complex plane, $\Re L(k) + i\Im L(k)$, leaving the negative half-axis, $\Re L(k) \le 0$, in the outer domain. This evidences that

$$\mathrm{Ind} L(k) = \frac{1}{2\pi}\left[\mathrm{Arg} L(\infty) - \mathrm{Arg} L(-\infty)\right] = 0. \tag{25}$$

It follows that $\gamma L(k) + 1 - \gamma$ possesses the same features at least for $0 \le \gamma \le 1$. Hence, in this range of γ

$$\mathrm{Ind}\mathcal{L}(k) = 0. \tag{26}$$

Further, $L(\pm\infty) = 1$ and it can be seen in (23) that this equality is still valid for $\mathcal{L}(k)$. The function $\mathcal{L}(k)$ can thus be factorized using the Cauchy-type integral, namely for $0 < \gamma < 1$

$$\mathcal{L}(k) = \mathcal{L}_+\mathcal{L}_-,$$
$$\mathcal{L}_\pm = \exp\left[\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln\mathcal{L}(\xi)}{\xi - k}d\xi\right], \tag{27}$$

where $\Im k > 0$ for $\mathcal{L}_+$ and vice versa. In particular, it follows that

$$\mathcal{L}_\pm \to 1 \quad (k \to \pm i\infty). \tag{28}$$

Noting that $\mathcal{L}(0) = 1/\gamma$ $[L(0) = \infty]$ we now can rearrange Eq. (21) in the form

$$\mathcal{L}_+(k)\Theta_+ + \frac{\Theta_-}{\mathcal{L}_-(k)} = \frac{Q^0[\mathcal{L}_+(k) - 1/\mathcal{L}_-(k)]}{0 + ik} = C_+(k) + C_-(k),$$
$$C_+(k) = \frac{Q^0[\mathcal{L}_+(k) - \mathcal{L}_+(0)]}{0 + ik}, \quad C_-(k) = \frac{Q^0[\mathcal{L}_+(0) - 1/\mathcal{L}_-(k)]}{0 + ik},$$
$$\mathcal{L}_\pm(0) = \frac{1}{\sqrt{\gamma}}\mathcal{R}^{\pm 1}, \quad \mathcal{R} = \exp\left[\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Arg}\,\mathcal{L}(\xi)}{\xi}d\xi\right]. \tag{29}$$

The functions $C_+(k)$ and $C_-(k)$ have no singular points in the upper and lower half-planes, respectively. Now, following the Wiener-Hopf technique, we can separate the corresponding functions in Eq. (29) and find a solution as follows:

$$\Theta_+ = \frac{C_+(k)}{\mathcal{L}_+(k)}, \quad \Theta_- = C_-(k)\mathcal{L}_-(k) \tag{30}$$

which for $\eta = 0$ yields

$$\begin{aligned}\Theta(0) = \lim_{p\to\infty} p\,\Theta_+(ip) = \lim_{p\to\infty} p\,\mathcal{T}_-(-ip) = Q^0\left[\mathcal{L}_+(0) - 1\right] \\ = Q^0\left(\frac{1}{\sqrt{\gamma}}\mathcal{R} - 1\right).\end{aligned} \tag{31}$$

In the considered wave, if it exists, $\Theta(0) = \Theta_c$. This presents an equation

$$Q^0\left(\frac{1}{\sqrt{\gamma}}\mathcal{R} - 1\right) = \Theta_c \tag{32}$$

which defines the wave speed. If no wave speed satisfies this equation the wave does not exist. This equation has a limiting expression for $\gamma \to 1$. In this case

$$\begin{aligned}\operatorname{Arg}\mathcal{L} = \arctan\frac{(1-\gamma)\Im L}{\gamma|L|^2 + (1-\gamma)\Re L} \sim (1-\gamma)\frac{\Im L}{|L|^2} = -(1-\gamma)\Im\left(\frac{1}{L}\right), \\ \mathcal{R} \sim 1 - \frac{1-\gamma}{\pi}\int_0^\infty \Im\left[\frac{1}{L(k)}\right]\frac{dk}{k}\end{aligned} \tag{33}$$

and in the limit Eq. (32) becomes

$$\frac{P^*}{\mu}\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Im\left[\frac{1}{L(k)}\right]\frac{dk}{k}\right\} = \Theta_c. \tag{34}$$

Below this result will be obtained directly for $\gamma = 1$.

The limits for $\eta \to \pm\infty$ follows from (30) [also see (22) and (29)] as

$$\begin{aligned}\Theta(+\infty) = \lim_{p\to 0} p\,\Theta_+(ip) = 0, \\ \Theta(-\infty) = \lim_{p\to 0} p\,\Theta_-(-ip) = \frac{P^*}{\gamma\mu}\end{aligned} \tag{35}$$

as it should be, namely, it is the static elongation corresponding to the first phase at $\eta > 0$ and the second phase at $\eta < 0$.

5 The case $\gamma = 1$

Such a bistable model is remarkable for the fact that the transition from the one branch to the other does not change the equations. It results only in self-equilibrated couples of fixed external forces applied to the particles connected by the corresponding bond, Figure 3. These forces are directed outwards to decrease the tensile force in accordance with the gap between the branches. Thus, an intact uniform lattice can be considered, and only the right-hand part of the dynamic equation (which is independent of the lattice displacements) reflects the transition. This simplifies greatly the problem considered. Namely, it now is not a mixed boundary value problem. Besides, in this case the superposition is valid allowing multiple line transitions to be considered. The elastic version of such a relation was introduced in Balk et al (2001a,b) with the use for the dynamics of a chain. Truskinovsky ???

In this case, Eq. (9) defines the elongation in an explicit form

$$\Theta^F(k) = \frac{P_* Q^F}{0+ik} = \frac{P_*}{\mu\,(0+ik)}\left[1 - \frac{1}{L(k)}\right]. \tag{36}$$

Since

$$\Re L(-k) = \Re L(k), \quad \Im L(-k) = -\Im L(k), \quad 1/L(0) = 0, \quad L(\pm\infty) = 1, \tag{37}$$

the inverse transform

$$\Theta(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Theta^F(k) e^{-ik\eta} dk \tag{38}$$

leads to

$$\Theta(0) = \frac{P_*}{\mu}\left[\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Im\left(\frac{1}{L(k)}\right)\frac{dk}{k}\right],$$

$$\Theta(\infty) = 0, \quad \Theta(-\infty) = \frac{P_*}{\mu} \tag{39}$$

which coincides with the limiting results, $\gamma \to 1$ [see Eqs. (34) and (35)].

5.1 Viscoelastic lattice

For this special type of the force-strain dependence, if the branches differ only by the 'elastic part' of the tensile force (as discussed below), the elastic-viscoelastic correspondence principle is valid.

In an initial range of the extension, the bonds are assumed now to satisfy the standard material viscoelastic relation:

$$T + \beta\frac{dT}{dt} = \mu\left(Q + \alpha\frac{dQ}{dt}\right), \tag{40}$$

where α and β are the creep and relaxation times, respectively. This relation corresponds to a unit presented in Figure 5, where the left spring stiffness is κ_1, the right spring stiffness is k_2 and the viscous resistance of the element successively connected with the right spring is c. In these terms

$$\mu = k_1, \quad \alpha = \frac{c}{\kappa_1} + \frac{c}{\kappa_2}, \quad \beta = \frac{c}{\kappa_2}. \tag{41}$$

Note that relation (40) can also be represented in the form

$$T = \frac{\mu}{\beta} e^{-t/\beta} \int_{-\infty}^{t} \left[Q(\tau) + \alpha \frac{dQ(\tau)}{d\tau} \right] e^{\tau/\beta} d\tau. \tag{42}$$

When the elongation reaches the critical value, Q_c, the left spring is assumed to slip by a fixed value, such that its elongation becomes $Q - Q_*$, where Q remains to be the elongation of the unit and

$$\kappa_1 Q_* = \mu\, Q_* = P_* = \text{ const.} \tag{43}$$

It is assumed that

$$P_* \leq \mu\, Q_c, \tag{44}$$

that is the tensile force in the left spring remains to be nonnegative. Thus, after this *transformation* the stress-strain relation (42) becomes

$$T = \frac{\mu}{\beta} e^{-t/\beta} \int_{-\infty}^{t} \left[Q(\tau) + \alpha \frac{dQ(\tau)}{d\tau} \right] e^{\tau/\beta} d\tau - P_*. \tag{45}$$

In accordance with the elastic-viscoelastic principle [or directly from (40)], the viscoelastic solution can be obtained from the elastic one (36) if μ is changed to the complex modulus $\mu\, \hat{E}$ with

$$\hat{E} = \frac{1 + ikV_\alpha}{1 + ikV_\beta}, \quad V_\alpha = \frac{v\alpha}{a}, \quad V_\beta = \frac{v\beta}{a}. \tag{46}$$

Accordingly, the function Y (16) must be replaced by

$$Y = \frac{(0 + ikV)^2}{\hat{E}}. \tag{47}$$

Formula (39) for $\Theta(0)$ becomes

$$\Theta(0) = \frac{P_*}{\mu} \left\{ \frac{\beta}{2\alpha} - \frac{1}{\pi} \int_0^\infty \Im \left[\frac{1}{\hat{E} L(k)} \right] \frac{dk}{k} \right\}. \tag{48}$$

5.2 Massless viscoelastic lattice

This model relates to a very low speed. If the inertia term is neglected, $L(k)$ becomes a real periodic function and the equality $L(\pm\infty) = 1$ in (37) does not hold. We should return to Eq. (36) which for the considered massless viscoelastic lattice becomes

$$\Theta^F(k) = \frac{P_*}{\mu\,(0 + ik)\hat{E}} \left[1 - \frac{1}{L(k)} \right] \tag{49}$$

with $Y = 0$. In this case, if $\beta > 0$ the elongation is discontinuous at $\eta = 0$ as follows directly from the behavior of the viscoelastic unit under discontinuous loading. The inverse transform leads to

$$\Theta(\pm 0) = \frac{P_*}{\mu}\left\{\frac{\beta}{2\alpha}\left[1 \mp \left(1 - < \frac{1}{L} >\right)\right]\right.$$
$$\left. + \left(1 - \frac{\beta}{\alpha}\right)\frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)L(k/V_\alpha)}\right\}, \tag{50}$$

where $< 1/L >$ is the averaged value of this periodic function. For the square-cell lattice it is

$$< \frac{1}{L} > = \frac{1}{\pi}\int_0^\pi \frac{\sin k/2}{\sqrt{1+\sin^2 k/2}}\, dk = \frac{1}{2}, \tag{51}$$

while for modes I and II triangular-cell lattice it is approximately equal to 1/3.

The integral in (50) can be asymptotically represented as

$$\frac{1}{\pi}\int_0^\infty \frac{dk}{(1+k^2)L(k/V_\alpha)} \to \left\{ \begin{array}{cc} 1/2 < 1/L > & (V_\alpha \to 0) \\ 0 & (V_\alpha \to \infty). \end{array}\right. \tag{52}$$

Regarding to the asymptote $V_\alpha \to \infty$ it is assumed that v remains well below the elastic wave speed, c, not to contradict the quasi-static formulation. This can be possible if $C_\alpha = \alpha c/a >>> 1$.

Note that the relation

$$\Theta(+0) = \Theta_c \tag{53}$$

must be used now. This relation has the following limiting expressions:

$$\Theta(+0) = \frac{P_*}{2\mu} < \frac{1}{L} > = \Theta_c \quad (V_\alpha = 0),$$
$$\Theta(+0) = \frac{\beta}{\alpha}\frac{P_*}{2\mu} < \frac{1}{L} > = \Theta_c \quad (V_\alpha = \infty). \tag{54}$$

It can be seen that the resistance to the considered localized wave propagation increases with its speed in an initial range of the ratio v/c. This increase is most pronounced for small ratios β/α. For $\beta = 0, \alpha > 0$ the resistance tends to infinity when $V_\alpha \to \infty$. Such a boundary layer type of the solution is similar to that obtained for the quasi-static crack growth in a viscoelastic lattice [Slepyan et al. (1999)].

This solution evidences that a stable region, where $Q(0)$ decreases as the speed increases, does exist if $\beta/\alpha < 1$. This conclusion, however, is based on the quasi-static formulation, which is valid if the viscosity times fall into the heart of the static-amplitude-response domain; that is, if $\alpha c/a$ is large enough, while β/Ga is small [see Slepyan (2000)].

In the elastic case, it follows that

$$\Theta(0) = \frac{P_*}{2\mu} < \frac{1}{L} > \tag{55}$$

and hence the quasi-static regime corresponds to the initial elongation

$$Q_0 = Q_c - \frac{P_*}{2\mu} < \frac{1}{L} > . \tag{56}$$

5.3 Energy release

The phase transformation path on the 'macrolevel' corresponds to the horizontal segment presented in Figure 6. The above analysis of the lattice shows explicitly how the energy barrier (triangle ABC) is overcome. Triangle CDE represents the released energy. The resulting energy released in this process per bond, represented by the difference of the areas of the latter and former triangles. For the quasi-static growth it is

$$G^0 \approx \frac{1}{6\mu}P_*^2 \quad \text{(modes I and II)},$$
$$G^0 = \frac{1}{4\mu}P_*^2 \quad \text{(mode III)}. \tag{57}$$

It should be noted that low-speed regime for an elastic lattice can be impossible. With regard to the crack growth this phenomenon is discussed in Marder and Gross (1995) and Slepyan (2000). However, under the influence of viscosity a stable growth within an initial region of the speed may become possible [Slepyan et al. (1999), Slepyan (2000)].

In a general case,

$$G^0 = \frac{\mu}{2}\left[\frac{1}{\gamma}\left(\frac{P_*}{\mu} - \Theta_c\right)^2 - \Theta_c^2\right]. \tag{58}$$

5.4 Continuous elastic model

The related steady-state problem for a continuous elastic medium can be formulated as that for the growing crack which faces interact with each other by a traction proportional to the crack opening and loaded by distributed external forces, p, — similar to those for lattices. We may base on the equation

$$L(k)[\sigma_+(k) + \sigma_-(k)] + u_-(k) = 0 \quad [u_+(k) = 0] \tag{59}$$

with

$$\sigma_- = \kappa u_- - \frac{p}{0 + ik} \tag{60}$$

and (for a subcritical crack speed)

$$L(k) = \frac{f(V)}{\mu\sqrt{k^2 + 0}}, \tag{61}$$

where $\sigma(\eta)$ is the traction acting on the upper elastic half-plane and $f(V)$ is a function of the nondimensional speed, V, different for different fracture modes (see, for example, Freund, 1990, or Slepyan, 1990). This leads to the equation

$$\mathcal{L}(k)\sigma_+ + u_- = \frac{\mathcal{L}(k)p}{0 + ik} \tag{62}$$

with

$$\mathcal{L}(k) = \frac{L(k)}{\kappa L(k) + 1}. \tag{63}$$

Using a standard way of considerations the Wiener-Hopf equation (63) can by resolved with the result as

$$\begin{aligned}
\sigma_+(k) &= \frac{1 - S_+(0)/S_+(k)}{0 + ik}, \quad u_-(k) = \frac{S_+(0)S_-(k)}{0 + ik}, \\
S_+(k) &= [\kappa - i\mu k/f(V)]^{-1/2} \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln S_0(\xi)}{\xi - k}\, d\xi\right] \quad (\Im k > 0), \\
S_-(k) &= [\kappa + i\mu k/f(V)]^{-1/2} \exp\left[-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln S_0(\xi)}{\xi - k}\, d\xi\right] \quad (\Im k < 0), \\
S_0(k) &= \frac{\sqrt{\kappa^2 + \mu^2 k^2/f^2(V)}}{\kappa + \mu\sqrt{k^2 + 0}/f(V)}, \\
S_\pm(0) &= \frac{1}{\sqrt{\kappa}}, \quad S_\pm(k) \sim \sqrt{\frac{f(V)}{\mu(0 \mp k)}} \quad (k \to \pm i\infty).
\end{aligned} \tag{64}$$

It follows that

$$\begin{aligned}
\sigma_+(k) &\sim \sqrt{\frac{\mu}{\kappa f(V)}}\frac{p}{\sqrt{0 - ik}} \quad (k \to i\infty), \\
u_-(k) &\sim \sqrt{\frac{f(V)}{\kappa\mu}}\frac{p}{(0 + ik)^{3/2}} \quad (k \to -i\infty), \\
u_-(k) &\sim \frac{p}{\mu(0 + ik)} \quad (k \to 0)
\end{aligned} \tag{65}$$

with the asymptotes of the original functions as

$$\begin{aligned}
\sigma(\eta) &\sim p\sqrt{\frac{\mu}{\pi\kappa f(V)\eta}} \quad (\eta \to +0), \\
u(\eta) &\sim 2p\sqrt{\frac{-f(V)\eta}{\pi\kappa\mu}} \quad (\eta \to -0), \\
u(\eta) &\to \frac{p}{\kappa} \quad (\eta \to -\infty).
\end{aligned} \tag{66}$$

The crack tip asymptotes in (66) result in the speed-independent energy release

$$G = \frac{p^2}{\kappa}. \tag{67}$$

This value is a half of the total work of the external forces; the other half is spent as the work against the resistance to the crack opening. This equality is valid here since, in contrast to the lattice model, no dissipative waves exist in the continuous model under a subcritical crack speed.

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Figure 1. The force—strain diagram. The dotted axes origin corresponds to the initial state of the lattice.

Figure 2. The lattices: (**a**) Square-cell lattice; (**b**) Triangular-cell lattice. The wave path bonds are shown by dotted lines.

Figure 3. External forces applied to the intact lattice to model the jump in the bond tensile force: (**a**) Square-cell lattice; (**b**) Triangular-cell lattice.

Figure 4. The force—strain diagram and the effective initial elongation.

Figure 5. The standard material unit.

Figure 6. The macrolevel phase transition path and the energy release.