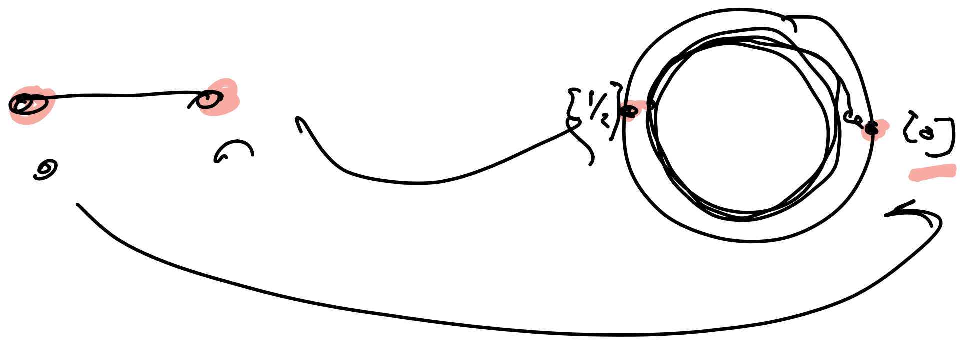
CLASS EXERCISE

Classify all pared homotopy classes

$f : ([0,1], \{0,1\}) \to (S^1, \{[0], [1/2]\})$.



What are the "model maps" all such f are homotopic to?

What are the f_n in this case?

To simplify assume that

$f(0) = [0]$, $f(1) = [1/2]$

$$\tilde{f}_n(t) = \left(n + \tfrac{1}{2}\right)t$$

$$f_n(t) = \left[\left(n + \tfrac{1}{2}\right)t\right] \qquad n \in \mathbb{Z}$$

$$f_n(0) = [n] = [0]$$

$$f_n(1) = \left[\tfrac{1}{2} + n\right] = [1/2]$$

$$\tilde{f}_n(t) = \left(n + \tfrac{1}{2}\right)t$$

EXISTENCE

Claim $\exists n$ s.t. $f \simeq f_n$

By LL. $\exists! \ \tilde{f} : [0,1] \to \mathbb{R}$

s.t. $\tilde{f}(0) = 0$ & $f = \pi \circ \tilde{f}$.

$\Rightarrow \quad f(1) = [1/2] = \pi \circ \tilde{f}(1)$

$\Rightarrow \quad \tilde{f}(1) \in \pi^{-1}([1/2])$

$\Rightarrow \exists! n$ s.t. $\tilde{f}(1) = 1/2 + n$.

Guess, that $f \simeq_p f_n$

$\tilde{f}_n(t) = (n + 1/2)\, t$

$\tilde{f}_n(1) = n + \frac{1}{2} = \tilde{f}(1)$

$\tilde{f}$ $\tilde{f}_n$

$$\tilde{G}(s,t) = (1-t)\,\tilde{f}(s) + t\,\tilde{f}_n(s)$$

homotopy between $\tilde{f}$ & $\tilde{f}_n$.

$$G(s,t) = \pi \circ \tilde{G}(s,t)$$

UNIQUENESS

Give $f : [0,1] \to S^1$

with $f(0) = [0]$ & $f(1) = [1/2]$

$\exists!\, n$ s.t. $f \simeq_p f_n$

PF strategy: Assume

$f \simeq_p f_n$ & $f \simeq_p f_m$

$\Rightarrow$ $f_n \simeq_p f_m$ $\Rightarrow$ $n = m$.

need to show

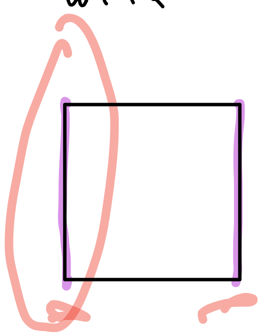
$\exists$ a homotopy

$$G : [0,1] \times [0.1] \to S^1$$

s.t. $f_1(t) = g_0(t) = G(0, t)$ &

$$f_2(t) = g_1(t) = G(1, t)$$

with $G([0,1] \times [0,1]) \subset \{[0], [1/2]\}$.



$G(0,0) = f_1(0) = [0]$

& $G(1,0) = f_2(1) = [1/2]$

$\Rightarrow G(\{0\} \times [0,1]) \subset \{[0]\}$

$G(\{1\} \times [0,1]) \subset \{[1/2]\}$

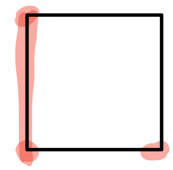
both connected so image must be connected.

By H.L.L $\exists!$

$$\tilde{G} : [0,1]^2 \to \mathbb{R}$$

with $\tilde{G}(0,0) = 0$ &

$G(s,t) = \pi \circ \tilde{G}(s,t)$.



Note $\tilde{g}_0(s) = \tilde{G}(s,0)$ is a lift of f_n with

$\tilde{g}_0(0) = 0 = \tilde{G}(0,0)$.

By the uniqueness in L.L $\Rightarrow$ $\tilde{g}_0 = \tilde{f}_n$

$\Rightarrow$ $\tilde{g}_0(1) = \tilde{f}_n(1) = n + 1/2$

We know

$\{[0]\} = G(\{0\} \times [0,1]) = \pi \circ \tilde{G}(\{0\} \times [0,1])$

discrete top.

↓

$\Rightarrow$ $\tilde{G}(\{0\} \times [0,1]) \subset \pi^{-1}([0]) = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Also $\tilde{G}(0,0) = 0$ → $\tilde{G}(\{0\} \times [0,1]) = \{0\}$

$\tilde{g}_1(s) = \tilde{G}(s,1)$ is a lift of f_m

& $\tilde{g}_1(0) = 0 \overset{\text{uniqueness of lifts}}{\Longrightarrow} \tilde{g}_1(s) = \tilde{f}_m(s)$

$\Rightarrow \tilde{f}_m(1) = m + 1/2$

same connectedness argument

$\Rightarrow \tilde{g}_0(1) = \tilde{g}_1(1)$

$$\begin{array}{ccc} \tilde{g}_0(1) & = & \tilde{g}_1(1) \\ \| & & \| \\ n + \frac{1}{2} & & m + \frac{1}{2} \end{array} \Rightarrow n = m.$$