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Directed intermittent search with stochastic resetting

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Abstract

We consider the directed intermittent search for one or more targets in a one-dimensional domain with stochastic resetting. A particle (searcher) randomly switches between a stationary search phase and a rightward moving ballistic phase. The particle can detect a target at some fixed rate whenever it is within range of the target and is in the stationary state. In the absence of resetting, there is a nonzero probability of failure. We calculate the hitting (detection) probability and conditional mean first passage time (MFPT) with and without resetting, for both a single target and a pair of competing targets. We also present an alternative probabilistic method for taking into account the effects of resetting, which is based on conditional expectations, stopping times and an application of the strong Markov property. Such an approach has previously been used to analyze diffusion processes in randomly switching environments.

Keywords: Stochastic resetting, random intermittent search, first passage times, strong Markov processes

(Some figures may appear in colour only in the online journal)

1. Introduction

A topic of increasing interest is the theory of stochastic processes under resetting [26]. The simplest example of such a process is a Brownian particle whose position is reset randomly in time at a constant rate r (Poissonian resetting) to some fixed point X_r, which could be its initial position. This system exhibits two of the major features observed in more complex models: (i) convergence to a nontrivial nonequilibrium stationary state (NESS); (ii) the mean time for a Brownian particle to find a hidden target is finite and has an optimal value as a function of the resetting rate r [17–19]. There have been numerous studies of more general stochastic processes with both Poissonian and non-Poissonian resetting, memory effects, and spatially extended systems, see the recent review [21] and references therein.

Of particular relevance to the analysis developed in this paper, is a study of a run-and-tumble particle under resetting [20]. In the absence of resetting such a model is an example of a one-dimensional velocity jump process

$$\frac{dX}{dt} = V_{N(t)}, \tag{1.1}$$

where $X(t)$ is the position of the particle at time t and the discrete random variable $N(t) \in \{1, \ldots, N\}$ indexes the current velocity state $V_{N(t)}$. Transitions between the velocity states are governed by a discrete Markov process with generator $\mathbf{A}$. Define $\mathbb{P}(x, n, t \mid y, m, 0)dx$ as the joint probability that $x \leqslant X(t) < x + dx$ and $N(t) = n$ given that initially the particle was at position $X(0) = y$ and was in state $N(0) = m$. Setting

$$p_n(x, t) \equiv \sum_m \mathbb{P}(x, t, n|0, 0, m)\sigma_m, \tag{1.2}$$

with initial condition $p_n(x, 0) = \delta(x)\sigma_n$, $\sum_m \sigma_m = 1$, the evolution of the probability density is described by the differential Chapman–Kolmogorov (CK) equation

$$\frac{\partial p_n}{\partial t} = -V_n \frac{\partial [p_n(x, t)]}{\partial x} + \sum_{n'=1}^{N} A_{nn'} p_{n'}(x, t). \tag{1.3}$$

In the run-and-tumble model considered in [20], there are two velocity states $V_1 = -v, V_2 = v$ and switching between the two states occurs at the same rate α. This symmetric model thus reduces to unbiased diffusion in the fast switching limit. The authors showed that when resetting is included, the system supports a NESS and that the time to reach a target can be optimized with respect to the resetting rate. One additional feature of the velocity jump process is that one needs to specify resetting protocols for both the position and velocity of the particle.

One-dimensional velocity-jump processes (without resetting) have been applied to a wide range of problems in cell biology [10], including run-and-tumble during bacterial chemotaxis [1, 2, 5, 23], molecular motor-based intracellular transport [9, 29], microtubule catastrophes [6, 15], and cytoneme-based morphogenesis [13]. They have also been used to model random intermittent search processes, where a particle switches between a set of ballistic non-search states and a stationary or slowly diffusing search state [4]; in the case of unbiased search and a single target in a bounded domain, the search time can be minimized with respect to the rates of switching between the different particle states.

In the presence of a chemotactic signal bacterial run-and-tumble becomes biased towards a food source (or away from a toxic region). This bias typically arises from a concentration-dependent modification in the switching rates between the different velocity states. More extreme cases of biased behavior can occur in other types of systems such as motor-driven intracellular transport [7, 28, 29]. For example, the directed transport of mRNA granules has been observed in the dendrites of neurons [16, 24, 32], see figure 1. First, mRNA-binding proteins allow mRNA to be sequestered away from cell nucleus by inhibiting translation. The repressed mRNAs are then packaged into ribonucleoprotein granules that are subsequently transported into the dendrite via molecular motor complexes. Finally, the mRNA is localized to an activated synapse by actin–based myosin motor proteins, and local translation is initiated following neutralization of the repressive mRNA-binding protein. By fluorescently labeling either the mRNA or mRNA-binding proteins and using live-cell imaging to track the movement of granules in cultured neurons, it has been found that under basal conditions, the majority of granules in dendrites are stationary or exhibits small oscillations around a few synaptic sites, whereas others exhibit bidirectional transport. However, activating a neuron

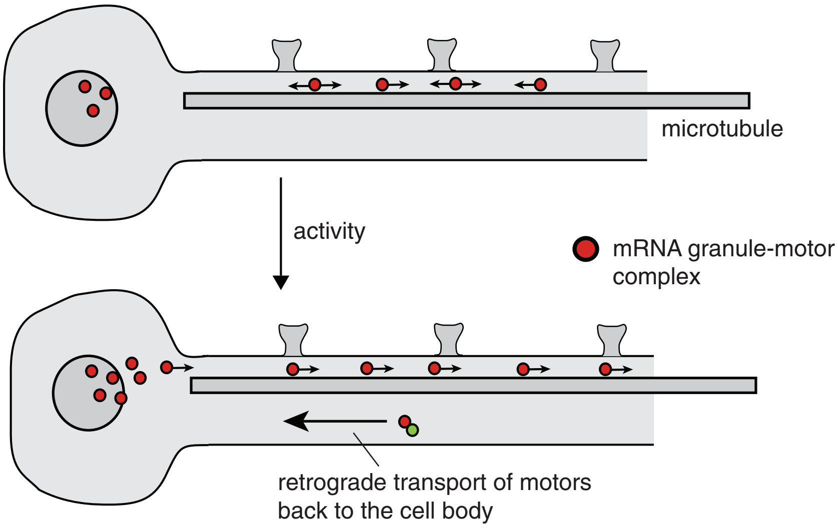

Figure 1. Schematic diagram illustrating mRNA granule mobility in dendrites. Under basal conditions, most granules are either stationary (or exhibit localized oscillations), whereas a minority exhibit bidirectional transport. Activity induces transcription of mRNA at the cell body and converts existing stationary granules into anterograde granules. A form of resetting could be based on the dynein dependent retrograde transport of motors back to the cell body.

using a chemical signal induces the conversion of stationary or oscillatory granules into predominantly anterograde-moving granules.

Motivated by the problem of mRNA transport, we previously analyzed a stochastic model of directed intermittent search for a hidden target on a semi-infinite track [7]. A particle injected at one end of the track randomly switches between a stationary search phase and a mobile, non-search phase that is biased in the anterograde direction. Since the particle has a non-zero probability of failure to find (be absorbed by) the target, it is not possible to optimize the search process in the sense of maximizing the target detection probability and minimizing the corresponding search time. On the other hand, if stochastic resetting to the origin were included in such a model, then the particle would eventually find the target and the search time could be minimized with respect to the resetting rate. Within the context of motor transport, it is known that molecular motors within a dendrite can be transported back to the cell body via the action of retrograde motors such as dynein. However, this retrograde transport will tend to have speeds of around 1 $\mu\text{m s}^{-1}$ so that resetting would not be instantaneous. Another interesting form of resetting could occur at the population level, since the removal and retrograde transport of motor complexes within the dendrite is supplemented by the injection of new motor complexes into the axon. If the rates of removal and injection were approximately balanced then there would be an effective form of resetting, but with more complicated statistics.

Another potential candidate for directed intermittent search with resetting is cytoneme-based morphogenesis [13]. Cytonemes are thin, dynamic, actin-rich cellular extensions with a diameter of around 100 nm and lengths that vary from 1 to 100 μm. There is growing experimental evidence that cytonemes can form direct cell-to-cell contacts, thus allowing the active transport of morphogens or their corresponding receptors to embryonic cells during development [22, 25, 33, 34]. In the particular case of Wnt signaling in zebrafish [34], it has been found that Wnt is clustered at the membrane tip of cytonemes, which nucleate from a source

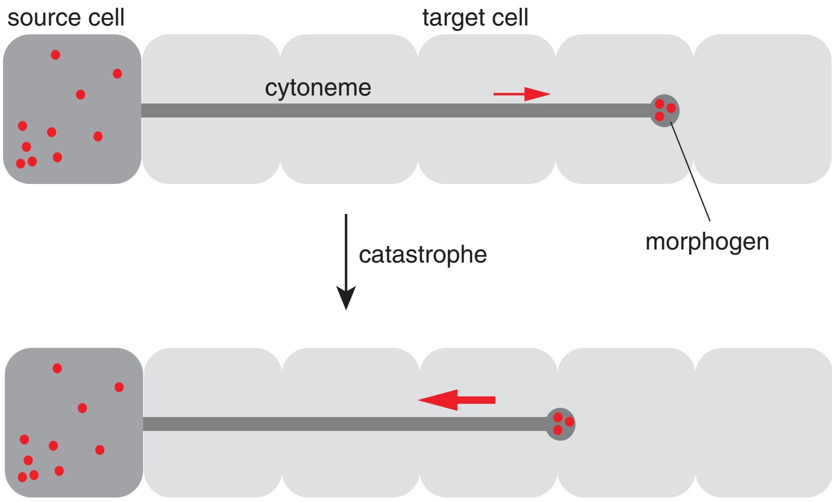

Figure 2. One-dimensional search-and-capture model of a cytoneme. For simplicity, the cytoneme grows along the surface of a one-dimensional array of target cells until it eventually forms a contact with one of the cells and delivers morphogen from its tip. There is a non-zero probability of the cytoneme undergoing a catastrophe and rapidly shrinking to zero (resetting), before a new search process begins.

cell and dynamically grow along the surface of an array of target cells, until making contact with one of the cells and delivering their cargo, see figure 2. However, analogous to microtubule catastrophes, a cytoneme can suddenly switch to a shrinkage phase and rapidly retract (reset). This is then followed by the growth of a newly nucleated cytoneme.

The above examples suggest that it is worthwhile exploring the general problem of directed intermittent search with stochastic resetting. In this paper, we focus on the limiting case of unidirectional transport, whereby a particle injected on to a semi-infinite track randomly switches between a stationary search phase and a rightward moving ballistic phase. (The particle could represent a molecular motor complex or the tip of a cytoneme, for example. However, the analysis is independent of any particular interpretation.) We begin by considering a single hidden target and calculating the hitting probability Π (probability of finding or being absorbed by the target) and the conditional MFPT T in the absence of resetting (section 2). Although these quantities have been determined previously [7], we use Laplace transform methods here in order to extend the analysis to include the effects of resetting. The latter is carried out in two ways. First, we write down a renewal equation for the survival probability in an analogous fashion to previous treatments [17, 18, 20]. Using Laplace transforms we show that the hitting probability is now unity and that the unconditional MFPT has a minimum value with respect to the resetting rate r. Second, in section 3 we show how the renewal method for stochastic resetting is equivalent to a probabilistic method based on conditional expectations, stopping times and an application of the strong Markov property. (For another approach to analyzing stochastic resetting without recourse to renewal theory, see [19, 30].) We have recently used such an approach to study a variety of problems, including search-and-capture models [13] and diffusion processes in randomly switching environments [11, 12]. The probabilistic method is then used to analyze the effects of resetting in the case of a pair of targets that compete for resources (section 4). For simplicity, we assume throughout that resetting occurs instantaneously after which the search process immediately restarts. However, one

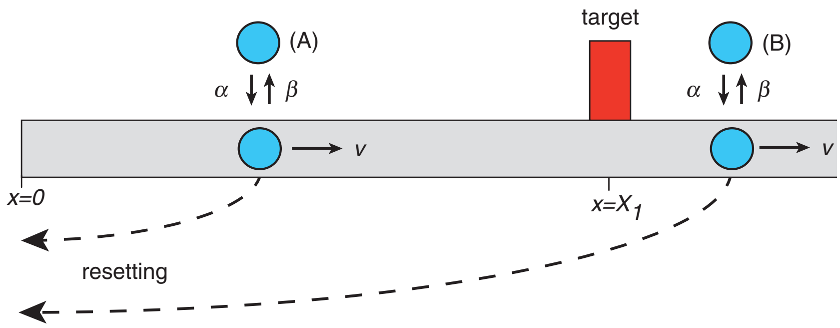

Figure 3. Stochastic model of directed intermittent search along a one-dimensional track. There is a target of fixed but unknown location at $x = X_1$. The particle is injected at $x = 0$ at time $t = 0$ and switches between a right-moving state with speed v and a stationary state. The particle can only find the target if it is in the stationary state and within a distance a of the target. In the absence of resetting, a particle at position (A) has a non-zero probability of finding the target, whereas a particle at position (B) cannot find the target. When resetting is included a particle at either position eventually finds the target.

could easily include some form of waiting time between successive search stages. In the case of cytonemes this could represent the time to nucleate a new cytoneme [13]. Finally,

2. Directed intermittent search for a single target

Consider a single particle moving along a semi-infinite, one-dimensional track, see figure 3. Suppose that at time $t = 0$ the particle enters one end of the track, which we take to be at $x = 0$. Within the interior of the track, $0 < x < \infty$, the particle can be in one of two states labeled by $n = 0, 1$: stationary ($n = 0$), or moving to the right (anterograde) with speed v. Transitions between the states are governed by a two-state Markov process. We further assume that there is a hidden target at a fixed but unknown location $x = X_1$. If the particle is within a distance a of the target and is in the stationary state, then the particle can detect or, equivalently, be absorbed by the target at a rate k. We assume throughout that $a < X_1$. This particular search problem was previously analyzed in [7]. Our goal is to determine how the search process is modified by the inclusion of stochastic resetting, whereby the particle can reset to its initial position $x = 0$ at a Poisson rate r, see figure 3. In order to proceed, it is first necessary to develop an alternative approach to analyzing the case without resetting.

2.1. Directed search without resetting

Let $X(t)$ and $N(t)$ denote the random position and state of the particle at time t. Setting

$$P_n(x,t|x_0,n_0)\mathrm{d}x = \mathbb{P}(x < X(t) < x + \mathrm{d}x, N(t) = n|X(0) = x_0, N(0) = n_0),$$

with initial condition $P_n(x,0|x_0,n_0) = \delta(x - x_0)\delta_{n,n_0}$, we have the following master equation describing the evolution of the probability densities for $t > 0$:

$$\frac{\partial P_1(x,t)}{\partial t} = -v\frac{\partial P_1(x,t)}{\partial x} - \beta P_1(x,t) + \alpha P_0(x,t) \qquad (2.1a)$$

$$\frac{\partial P_0(x,t)}{\partial t} = \beta P_1(x,t) - \alpha P_0(x,t) - k\chi(x - X_1)P_0(x,t), \tag{2.1b}$$

where we have dropped the explicit dependence on initial conditions. Here α, β are the transition rates between the stationary and mobile state as indicated in figure 3. We have introduced the indicator function χ according to

$$\chi(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Since the particle cannot return to the origin (without resetting), it is not necessary to impose any boundary conditions.

Let $J(t)$ denote the probability flux due to absorption by the target at X given that $x_0 = 0$ and $n_0 = 1$:

$$J(t) = k\int_{X-a}^{X+a} P_0(x,t)\mathrm{d}x. \tag{2.3}$$

Define the hitting probability Π to be the probability that the particle eventually finds the target, that is, it is absorbed somewhere in the interval $X - a \leqslant x \leqslant X + a$ rather than irreversibly passing beyond the target:

$$\Pi = \int_0^\infty J(t)\mathrm{d}t. \tag{2.4}$$

The conditional mean first passage time (MFPT) T is then defined to be the mean time it takes for the particle to find the target given that it does find the target:

$$T = \frac{\int_0^\infty tJ(t)\mathrm{d}t}{\int_0^\infty J(t)\mathrm{d}t}. \tag{2.5}$$

2.2. Calculation of Π and T

There are two alternative methods for calculating the hitting probability Π and the conditional MFPT T defined by equations (2.4) and (2.5), one based on the forward master equation and the other based on solving the corresponding backward equation. Previously we used the latter approach by working with probability fluxes in the time domain [7]. Here we will use Laplace transforms and survival probabilities so that the analysis can be extended to include resetting.

Introduce the survival probability $Q_m(x_0,t)$ that the particle has not yet been absorbed by the target at time t given that it started at $x = x_0$ in state m. That is,

$$Q_m(x_0,t) = \int_0^\infty [P_0(x,t|x_0,m) + P_1(x,t|x_0,m)]\mathrm{d}x. \tag{2.6}$$

Note that $Q_m(x_0,t) = 1$ for $x_0 > X_1 + a$. Setting $m = 1$, $x_0 = 0$, and $P_n(x,t) = P_n(x,t|0,1)$, we have

$$\begin{aligned}\frac{\partial Q_1(0,t)}{\partial t} &= \int_0^\infty \frac{\partial[P_0(x,t)+P_1(x,t)]}{\partial t}\mathrm{d}x \\ &= \int_0^\infty \left[-v\frac{\partial P_1(x,t)}{\partial x} - k\chi(x-X_1)P_0(x,t)\right]\mathrm{d}x \\ &= -k\int_{X_1-a}^{X_1+a} P_0(x,t)\mathrm{d}x = -J(t),\end{aligned}$$

where $J(t)$ is the probability flux into the target. It follows that

$$\Pi = 1 - \lim_{t\to\infty} Q_1(0,t), \quad T = \Pi^{-1}\int_0^\infty [Q_1(0,t) - (1-\Pi)]\mathrm{d}t. \tag{2.7}$$

The survival probabilities evolve according to the backward master equation

$$\frac{\partial Q_1(x_0,t)}{\partial t} = v\frac{\partial Q_1(x_0,t)}{\partial x_0} - \beta[Q_1(x_0,t) - Q_0(x_0,t)], \tag{2.8a}$$

$$\frac{\partial Q_0(x_0,t)}{\partial t} = \alpha[Q_1(x_0,t) - Q_0(x_0,t)] - k\chi(x_0 - X_1)Q_0(x_0,t), \tag{2.8b}$$

for $0 < x_0 \leqslant X_1 + a$. Introducing the Laplace transforms

$$\widetilde{Q}_m(x_0,s) = \int_0^\infty \mathrm{e}^{-st}Q_m(x_0,t)\mathrm{d}t, \tag{2.9}$$

and using the initial conditions $Q_m(x_0,0) = 1$, we have

$$v\frac{\partial \widetilde{Q}_1(x_0,s)}{\partial x_0} - (\beta+s)\widetilde{Q}_1(x_0,s) + \beta\widetilde{Q}_0(x_0,s) = -1, \tag{2.10a}$$

$$\alpha\widetilde{Q}_1(x_0,s) - (\alpha+s)\widetilde{Q}_0(x_0,s) - k\chi(x_0-X_1)\widetilde{Q}_0(x_0,s) = -1, \tag{2.10b}$$

with $\widetilde{Q}_1(x_0,s) = 1/s$ for $x_0 > X_1 + a$. Solving equation (2.10*b*) for $\widetilde{Q}_0$ gives

$$\widetilde{Q}_0(x_0,s) = \frac{\alpha\widetilde{Q}_1(x_0,s) + 1}{\alpha + s + k\chi(x_0 - X_1)}. \tag{2.11}$$

Substituting (2.11) into equation (2.10*a*) then yields

$$\frac{\partial \widetilde{Q}_1(x_0,s)}{\partial x_0} = \frac{1}{v}\left(\beta + s - \frac{\alpha\beta}{\alpha+s+k\chi}\right)\widetilde{Q}_1(x_0,s) - \frac{1}{v}\left(\frac{\beta}{\alpha+s+k\chi}+1\right). \tag{2.12}$$

This equation can be solved separately in the three domains $0 < x < X_1 - a$, $X_1 - a < x < X_1 + a$ and $X_1 + a < x < \infty$, after imposing continuity across the boundaries at $x = X_1 \pm a$ together with the condition $\widetilde{Q}_1(x_0,s) = 1/s$ for $x_0 \geqslant X_1 + a$. It is convenient to set

$$f_k(s) = \frac{1}{v}\left(\beta + s - \frac{\alpha\beta}{\alpha+s+k}\right), \quad g_k(s) = \frac{1}{v}\left(\frac{\beta}{\alpha+s+k}+1\right).$$

In the first domain, the solution has the form

$$\widetilde{Q}_1(x_0,s) = A(s)\mathrm{e}^{f_0(s)x_0} - \frac{g_0(s)}{f_0(s)}\left(\mathrm{e}^{f_0(s)x_0} - 1\right), \tag{2.13}$$

and in the second domain it is given by

$$\widetilde{Q}_1(x_0, s) = \frac{1}{s} - \left[\frac{1}{s} - \frac{g_k(s)}{f_k(s)}\right]\left(1 - \mathrm{e}^{f_k(s)[x - X_1 - a]}\right). \tag{2.14}$$

The latter satisfies the continuity condition at $x = X_1 + a$. Imposing continuity at $x = X_1 - a$ then determines the constant A according to

$$A(s) = \mathrm{e}^{-f_0(s)[X_1 - a]}\left\{\frac{g_0(s)}{f_0(s)}\left(\mathrm{e}^{f_0(s)[X_1 - a]} - 1\right) + \frac{\mathrm{e}^{-2af_k(s)}}{s} + \frac{g_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{-2af_k(s)}\right)\right\}. \tag{2.15}$$

It is now straightforward to read off the hitting probability and conditional MFPT. In terms of Laplace transforms,

$$\Pi = 1 - \lim_{s\to 0} s\widetilde{Q}_1(0, s) = 1 - \lim_{s\to 0} sA(s) = 1 - \mathrm{e}^{-2f_k(0)a} \tag{2.16}$$

and

$$\begin{aligned} T &= \Pi^{-1}\lim_{s\to 0}\left(\widetilde{Q}_1(0, s) - \frac{1-\Pi}{s}\right) \\ &= \Pi^{-1}\lim_{s\to 0}\left[\mathrm{e}^{-f_0(s)[X_1 - a]}\left\{\frac{g_0(s)}{f_0(s)}\left(\mathrm{e}^{f_0(s)[X_1 - a]} - 1\right) + \frac{g_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{-2af_k(s)}\right)\right.\right. \\ &\qquad \left.\left. + \frac{\mathrm{e}^{-2af_k(s)}}{s}\right\} - \frac{\mathrm{e}^{-2af_k(0)}}{s}\right] \\ &= \Pi^{-1}\left\{\left(g_0(0) - f_0'(0)\mathrm{e}^{-2af_k(0)}\right)[X_1 - a] + \frac{g_k(0)}{f_k(0)}\left(1 - \mathrm{e}^{-2af_k(0)}\right)\right. \\ &\qquad \left. - 2af_k'(0)\mathrm{e}^{-2af_k(0)}\right\}. \end{aligned} \tag{2.17}$$

Setting

$$\lambda = f_k(0) = \frac{\beta}{v}\frac{k}{\alpha + k}, \quad \mu_1 = g_k(0) = \frac{\beta + \alpha}{v\alpha} \tag{2.18}$$

and

$$\mu_2 = f_k'(0) = \frac{1}{v}\left(1 + \frac{\alpha\beta}{(\alpha + k)^2}\right), \quad \mu_3 = \frac{g_k(0)}{f_k(0)} = \frac{\alpha + \beta + k}{\beta k}, \tag{2.19}$$

we obtain the result

$$\Pi = 1 - \mathrm{e}^{-2\lambda a}, \quad T = (X_1 - a)\mu_1 - \frac{2a\mu_2}{\mathrm{e}^{2\lambda a} - 1} + \mu_3. \tag{2.20}$$

The latter is identical to the one obtained using alternative methods [7].

In figure 4 we plot Π and T as functions of α for different values of β. It can be seen that increasing the parameter α, which controls how much time the particle spends in the stationary search mode, decreases both the hitting probability and the conditional MFPT. Similarly, increasing the parameter β, which controls how much time the particle spends in the anterograde mobile state, increases both the hitting probability and the MFPT. Our results are consistent with the intuitive picture that one cannot simultaneously maximize the hitting probability and minimize the MFPT. Note that since we do not have any specific application in mind, we fix time and space units by setting $k = 1$ and $a = 1$, respectively.

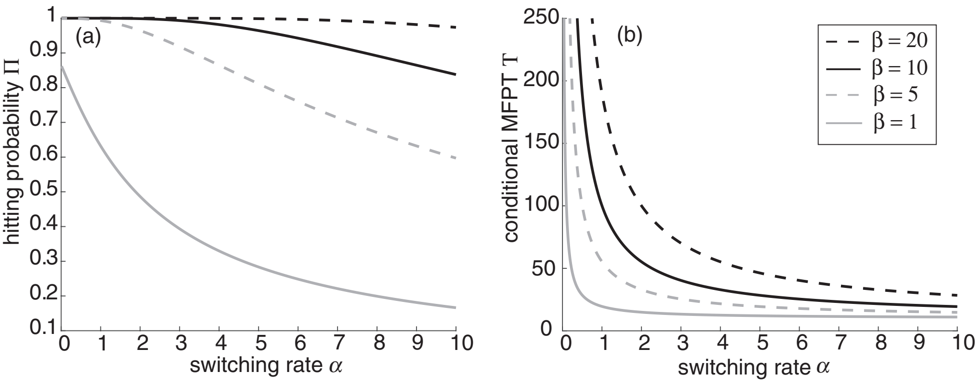

Figure 4. Plots of (a) the hitting probability Π and (b) the MFPT T as a function of α for different β. Other parameter values are $X = 10$, $a = 1$, $k = 1$, and $v = 1$.

2.3. Directed search with stochastic resetting

Now suppose that the particle can reset to its initial position $x = x_0$ and state $n_0 = 1$ at a constant rate r (Poissonian resetting), see figure 3. We will take $x_0 = 0$ below. (With a small modification, one could assume instead that the particle resets to the stationary state or a mixture of the two, see below.) Since the resetting process preserves the initial conditions, we have the renewal equation

$$Q_1^{(r)}(x_0, t) = \mathrm{e}^{-rt} Q_1(x_0, t) + r \int_0^t \mathrm{e}^{-r\tau} Q_1(x_0, \tau) Q_1^{(r)}(x_0, t - \tau) \mathrm{d}\tau, \tag{2.21}$$

where $Q_m^{(r)}(x_0, t)$ is the survival probability in the presence of resetting and $Q_m(x_0, t)$ is the survival probability without resetting, as calculated in section 2.2. The first term on the right-hand side represents trajectories with no resettings. The integrand in the second term is the contribution from trajectories that last reset at time $\tau \in (0, t)$, and consists of the product of the survival probability starting from x_0 with resetting up to time $t - \tau$ and the survival probability starting from X_r without any resetting for the time interval τ. Since we have a convolution, it is natural to introduce the Laplace transform

$$\widetilde{Q}_1(x_0, s) = \int_0^\infty Q_1(x_0, t) \mathrm{e}^{-st} \mathrm{d}t,$$

and similarly for $\widetilde{Q}_1^{(r)}(x_0, s)$. Laplace transforming the renewal equation and rearranging leads to the general result

$$\widetilde{Q}_1^{(r)}(x_0, s) = \frac{\widetilde{Q}_1(x_0, r + s)}{1 - r\widetilde{Q}_1(x_0, r + s)}. \tag{2.22}$$

Two results immediately follow from this renewal condition with $x_0 = 0$. First, the hitting probability with resetting is given by

$$\Pi^{(r)} = 1 - \lim_{t\to\infty} Q_1^{(r)}(0,t) = 1 - \lim_{s\to 0} s\widetilde{Q}_1^{(r)}(0,s)$$
$$= 1 - \lim_{s\to 0} s \frac{\widetilde{Q}_1(0,r+s)}{1 - r\widetilde{Q}_1(0,r+s)}$$
$$= 1 - \lim_{s\to 0} s \frac{A(r)}{1 - rA(r)} = 1. \tag{2.23}$$

That is, in contrast to the previous case without resetting, the particle does eventually find the target. Second, the now unconditional MFPT takes the form

$$T^{(r)} = \int_0^\infty Q_1^{(r)}(0,t)\mathrm{d}t = \widetilde{Q}_1^{(r)}(0,0) = \frac{A(r)}{1 - rA(r)}. \tag{2.24}$$

Note from equation (2.11) that if we had taken the initial and reset state to be the stationary state $n_0 = 0$ rather then the ballistic state, then

$$T^{(r)} = \frac{A(r)}{1 - rA(r)} + \frac{1}{\alpha},$$

where α^{-1} is the mean time to exit the stationary state. (It would also be straightforward to include a more general waiting time density for the time to enter the ballistic state at the origin.)

We now observe that $T^{(r)} \to \infty$ as $r \to 0$, since in the no resetting limit there is a nonzero probability of failure to find the target, and these particular sample paths have an infinite FPT. Morever, $T^{(r)} \to \infty$ in the limit $r \to \infty$, as resetting occurs so often that the particle never has the chance to reach the target. Analogous to Brownian motion with stochastic resetting [17, 18], this suggests that there is a value of r that minimizes $T^{(r)}$. The latter is indeed found to be the case, as demonstrated in figures 5 and 6. However, the dependence on the switching rates α, β is also non-trivial. For fixed $\beta = 10$, say, we know from figure 4 that in the absence of resetting the hitting probability is high for small α and decreases as α increases. This suggests that the optimal rate of resetting should be an increasing function of α, in order to counteract the increasing number of failures, see figure 5. Moreover, for small α the MFPT $T^{(r)}$ increases with r, whereas for large α it decreases with r, see figure 7. Now suppose that $\alpha = 10$, say, and we vary β. Figure 4 shows that the hitting probability decreases significantly as β decreases (no resetting), suggesting that optimal resetting rate increases, which is seen in figure 6. Finally, note that an upper bound on the optimal resetting rate will depend on the ratio v/X_1, since there is no advantage resetting before a particle can physically reach the target. For the given parameter values $v/X_1 = 0.1$.

3. The renewal method and the strong Markov property

Implicit in the use of renewal methods is that the underlying stochastic process satisfies the strong Markov property. In order to explore this connection further, we briefly review some probability theory. First, it is useful to introduce the notion of a natural filtration. Given a stochastic process $\{X(t) : t \in \Sigma]\}$ on some time interval Σ, the natural filtration $\mathcal{F}_t$ is taken to be the set of sample paths generated by the stochastic process, that is,

$$\mathcal{F}_t = \{X(s), s \leqslant t\}.$$

(More precisely, $\mathcal{F}_t$ belongs to a σ-algebra.) In other words, $\mathcal{F}_t$ contains the history of the stochastic process up to time t. A random variable is said to be measurable with respect to the

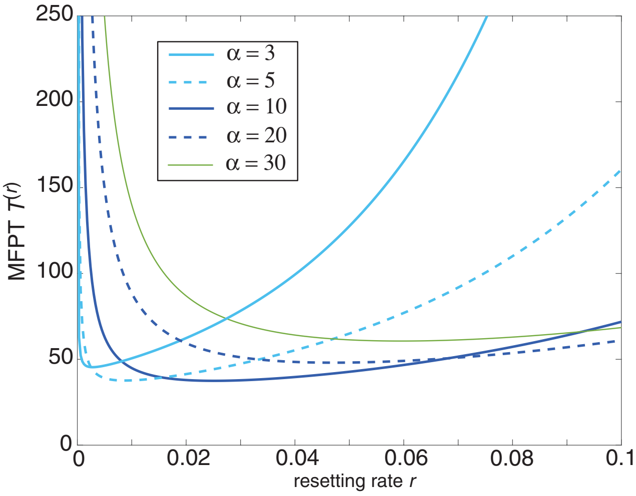

Figure 5. Plots of the MFPT $T^{(r)}$ as a function of the resetting rate r for various switching rates α with $k = v = 1$, $\beta = 10$, $X_1 = 10$ and $a = 1$.

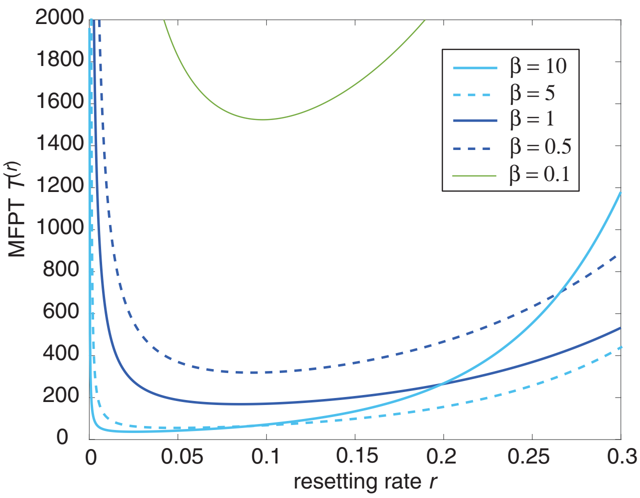

Figure 6. Plots of the MFPT $T^{(r)}$ as a function of the resetting rate r for various switching rates β with $k = v = 1$, $\alpha = 10$, $X_1 = 10$ and $a = 1$.

natural filtration $\mathcal{F}_t$ if its probability density can be determined from the distribution of sample paths up to time t. A random variable $\mathcal{T} \in \mathbb{R}^+$, which is defined on the same probability space as $X(t)$, is a stopping time if for every $t \in \Sigma$ we can completely determine whether or not $\mathcal{T}$ has occurred before time t using the information contained in all the events that have occurred up to time t, that is, the filtration $\mathcal{F}_t$. A classical example of a stopping time is a first passage time.

Recall that a stochastic process $\{X(t)\}_{t\in\Sigma}$ is said to have the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it. That is, for all $t' > t$ we have

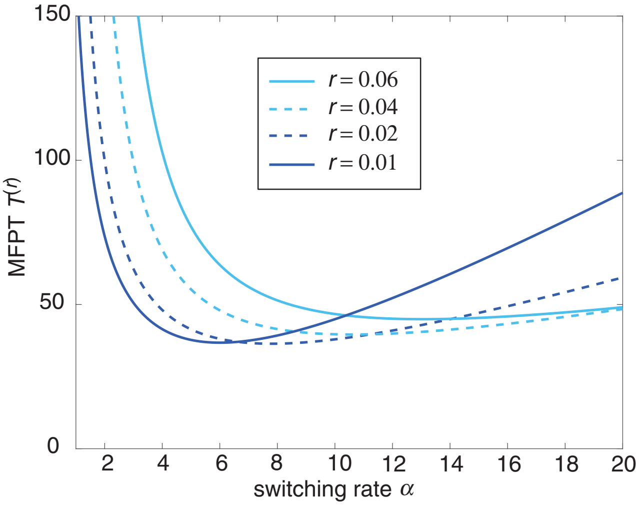

Figure 7. Plots of the MFPT $T^{(r)}$ as a function of the switching rate α for various resetting rates r with $k = v = 1$, $\beta = 10$, $X_1 = 10$ and $a = 1$.

$$\mathbb{P}[X_{t'} \leqslant x | X_s, s \leqslant t] = \mathbb{P}[X_{t'} \leqslant x | X_t].$$

The strong Markov property is similar to the Markov property, except that the 'present' is defined in terms of a stopping time. That is, given any finite-valued stopping time $\mathcal{T}$ with respect to the natural filtration of X, if the stochastic process $Y(t) = X(t + \mathcal{T}) - X(\mathcal{T})$ is independent of $\{X(s), s < \mathcal{T}\}$ and has the same distribution as $\widehat{Y}(t) = X(t) - X(0)$ then X is said to satisfy the strong Markov property.

In order to apply the above ideas to stochastic resetting, we introduce two new discrete random variables. First, let $L(t) \in \{0, 1\}$ indicate whether the particle has $(L(t) = 1)$ or has not $(L(t) = 0)$ been absorbed by the target in the time interval $[0, t]$. Second, let $N_{\text{reset}}(t)$ be the Poisson process associated with resetting, which counts the number of resets up to time t. It follows that

$$\mathbb{P}[N_{\text{reset}}(t) = n | N_{\text{reset}}(t) = 0] = \frac{r^n \mathrm{e}^{-rt}}{n!}. \tag{3.1}$$

Consider the following set of first passage times;

$$\mathcal{T} = \inf\{t > 0; X_1 - a \leqslant X(t) \leqslant X_1 + a, L(t) = 1\},$$
$$\mathcal{S} = \inf\{t > 0; N_{\text{reset}}(t) = 1, L(t) = 0\},$$
$$\mathcal{R} = \inf\{t > 0; X_1 - a \leqslant X(t + \mathcal{S}) \leqslant X_1 + a, L(t + \mathcal{T}_0) = 1\},$$

where we have suppressed the explicit dependence on the initial condition $X(0) = 0, N(0) = 1$. Here $\mathcal{T}$ is the FPT for finding the target irrespective of the number of resettings, $\mathcal{S}$ is the FPT for the first resetting given that the target hasn't yet been found, and $\mathcal{R}$ is the FPT for finding the target given that at least one resetting has occurred. Next we introduce the sets

$$\Omega = \{\mathcal{T} < \infty\}, \quad \Gamma = \{\mathcal{S} < \mathcal{T} < \infty\} \subset \Omega.$$

That is, Ω is the set of all events for which the particle is eventually absorbed by the target, and Γ is the subset of events in Ω for which the particles resets at least once. It immediately follows that

$$\Omega\backslash\Gamma = \{\mathcal{T} < \mathcal{S} = \infty\}.$$

In other words, $\Omega\backslash\Gamma$ is the set of all events for which the particle is absorbed without any resetting. We now use probabilistic arguments to recalculate the splitting probability $\Pi^{(r)}$ and MFPT $T^{(r)}$ in the presence of resetting.

First, the hitting probability $\Pi^{(r)}$ can be decomposed as

$$\Pi^{(r)} := \mathbb{P}[\Omega] = \mathbb{P}[\Omega\backslash\Gamma] + \mathbb{P}[\Gamma]. \tag{3.2}$$

We note that

$$\mathbb{P}[\Omega\backslash\Gamma] = \mathbb{P}[\mathcal{T}1_{\Omega\backslash\Gamma} < \infty] = -\int_0^\infty \mathrm{e}^{-r\tau}\frac{\mathrm{d}Q_1(0,\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = -r\widetilde{Q}_1(0,r) + 1.$$

We also have

$$\mathbb{P}[\Gamma] = \mathbb{P}[\mathcal{S} < \infty]\mathbb{P}[\mathcal{R} < \infty]$$

with

$$\mathbb{P}[\mathcal{S} < \infty] = \int_0^\infty rQ_1(0,\tau)\mathrm{e}^{-r\tau}\mathrm{d}\tau = r\widetilde{Q}_1(0,r).$$

Finally, from the strong Markov property $\mathbb{P}[\mathcal{R} < \infty] = \Pi^{(r)}$, so that

$$\Pi^{(r)} = 1 - r\widetilde{Q}_1(0,r) + r\widetilde{Q}_1(0,r)\Pi^{(r)},$$

which implies that $\Pi_r = 1$.

Similarly, we can decompose the unconditional MFPT $T^{(r)} = \mathbb{E}[\mathcal{T}]$ as

$$T^{(r)} = \mathbb{E}[\mathcal{T}1_{\Omega\backslash\Gamma}] + \mathbb{E}[\mathcal{T}1_{\Gamma}], \tag{3.3}$$

with

$$\mathbb{E}[\mathcal{T}1_{\Omega\backslash\Gamma}] = -\int_0^\infty \tau\mathrm{e}^{-r\tau}\frac{\mathrm{d}Q_1(0,\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \left[1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right]\widetilde{Q}_1(0,r),$$

and

$$\mathbb{E}[\mathcal{T}1_{\Gamma}] = \mathbb{E}[(\mathcal{S}+\mathcal{R})1_{\Gamma}] = \mathbb{E}[\mathcal{S}1_{\Gamma}] + \mathbb{E}[\mathcal{R}1_{\Gamma}]. \tag{3.4}$$

We now note that

$$\mathbb{E}[\mathcal{S}1_{\Gamma}] = r\int_0^\infty \tau Q_1(0,\tau)\mathrm{e}^{-r\tau}\mathrm{d}\tau = -r\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{Q}_1(0,r),$$

and from the strong Markov property,

$$\mathbb{E}[\mathcal{R}1_{\Gamma}] = T^{(r)}\mathbb{P}[\Gamma] = T^{(r)}r\widetilde{Q}_1(0,r).$$

Combining all of these results,

$$T^{(r)} = \left[1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right]\widetilde{Q}_1(0,r) - r\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{Q}_1(0,r) + T^{(r)}r\widetilde{Q}_1(0,r),$$

which on rearranging recovers equation (2.24).

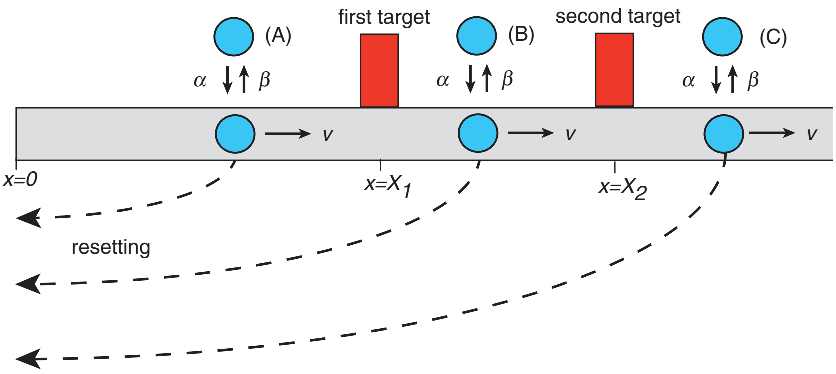

Figure 8. Schematic diagram of unidirectional search with a pair of targets at X_1 and X_2, respectively. In the absence of resetting, only the particle at (A) has a non-zero probability of finding either target. The particle at position (B) only has a chance of finding the second target, whereas the particle at position (C) fails to detect either target. When resetting is included, all three particles eventually find one of the targets.

4. Pair of targets and competition

So far we have considered anterograde intermittent search for a single hidden target, under the assumption that additional targets are further downstream. In the absence of resetting, downstream targets have no effect on the hitting probability and conditional MFPT of the upstream target. However, an upstream target will have a significant effect on the search for downstream targets. In order to explore this phenomenon, let us consider two identical targets at positions $x = X_1$ and $x = X_2$ where the distance between the two targets is $X_2 - X_1 > 2a$, see figure 8. We begin by considering the case without resetting by extending the analysis of section 2. Again this provides an alternative derivation to the one developed in [7], which is more suitable for incorporating stochastic resetting.

4.1. Pair of targets without resetting

Equation (2.13) establishes that the hitting probability $\pi_1(x_0)$ to find the first target when $x_0 < X_1 - a$ is

$$\pi_1(x_0) = 1 - \lim_{s\to 0} s\widetilde{Q}_1(x_0, s) = 1 - \mathrm{e}^{-2\lambda a} = \Pi,$$

with λ given by equation (2.18). That is, the hitting probability is independent of the starting position x_0 provided that it is to the left of the target detection domain $[X_1 - a, X_1 + a]$. It immediately follows that the probability that the particle will miss the first target and then find the second target is simply

$$\Pi_2 = (1 - \Pi)\Pi. \tag{4.1}$$

Note that a similar argument can be applied to multiple targets. In particular, suppose that there are M targets at positions X_l, $l = 1, \ldots, M$, with $X_1 < X_2 < \ldots < X_M$ and no overlap between the reaction domains. The probability of missing the first n targets and finding the $(n + 1)$th target can be formulated in terms of a series of Bernouilli trials that have probability Π of success and $1 - \Pi$ of failure. In particular, $\Pi_{n+1} = (1 - \Pi)^n\Pi$. Hence, $\Pi_{n+1} = \mathrm{e}^{-2na\lambda} - \mathrm{e}^{-2(n+1)a\lambda}$.

The condition $\partial\Pi_{n+1}/\partial\beta = 0$ then implies that the maximum hitting probability is achieved when α and β satisfy the linear relationship

$$\beta_{\max}(\alpha) = \frac{\ln\left(1+\frac{1}{n}\right)v}{2ka}(\alpha + k). \tag{4.2}$$

The maximum hitting probability of the $(n+1)$th target is thus

$$\Pi_{\max} = \frac{n^n}{(n+1)^{n+1}}. \tag{4.3}$$

Interestingly the maximum hitting probability depends only on the number of targets.

In order to determine the conditional MFPTs, we introduce the discrete stochastic variable $L(t) \in \{0, 1, 2\}$ with $L(t) = l$, $l = 1, 2$, indicating that the particle has been absorbed by the lth target before time t and $L(t) = 0$ indicating that the particle is still free. (In the case of M targets we would take $L(t) \in \{0, 1, \ldots M\}$.) Consider the FPT

$$T_{l,m}(x_0) = \inf\{t \geqslant 0; X_l - a \leqslant X(t) \leqslant X_l + a, L(t) = l | X(0) = x_0, N(0) = m\}.$$

The probability flux into the lth target cell is

$$J_{l,m}(x_0, t) = k\int_{X_l-a}^{X_l+a} P_0(x, t|x_0, m)\mathrm{d}x,$$

so that

$$\Pi_{l,m}(x_0, t) := \mathbb{P}[0 < T_l^m(x_0) < t] = \int_0^t J_{l,m}(x_0, t')\mathrm{d}t',$$

where $\Pi_{l,m}(x_0, t)$ is the probability that the particle is captured by the lth target before time t, having started in state (x_0,m). Since

$$\frac{\mathrm{d}\Pi_{l,m}(x_0, t)}{\mathrm{d}t} = J_{l,m}(x_0, t) = \int_0^t \frac{\mathrm{d}J_{l,m}(x_0, t')}{\mathrm{d}t'}\mathrm{d}t' + J_{l,m}(x_0, 0),$$

and $J_{l,m}(x_0, 0) = k\chi(x_0 - X_l)\delta_{m,0}$, it follows that $\Pi_{l,m}(x_0, t)$ satisfies the backward equation

$$\frac{\partial\Pi_{l,1}(x_0, t)}{\partial t} = v\frac{\partial\mathbb{P}_{l,1}(x_0, t)}{\partial x_0} - \beta[\Pi_{l,1}(x_0, t) - \Pi_{l,0}(x_0, t)], \tag{4.4a}$$

$$\frac{\partial\Pi_{l,0}(x_0, t)}{\partial t} = \alpha[\Pi_{l,1}(x_0, t) - \Pi_{l,0}(x_0, t)] - k\sum_{j=1,2}\chi(x_0 - X_j)\Pi_{l,0}(x_0, t) + k\chi(x_0 - X_l)\delta_{m,0}, \tag{4.4b}$$

with the initial conditions $\Pi_{l,m}(x_0, 0) = 0$. Laplace transforming equations (4.4a) and (4.4b) yields

$$v\frac{\partial\widetilde{\Pi}_{l,1}(x_0, s)}{\partial x_0} - (\beta + s)\widetilde{\Pi}_{l,1}(x_0, s) + \beta\widetilde{\Pi}_{l,0}(x_0, s) = 0, \tag{4.5a}$$

$$\alpha\widetilde{\Pi}_{l,1}(x_0, s) - (\alpha + s)\widetilde{\Pi}_{l,0}(x_0, s) - k\sum_{j=1,2}\chi(x_0 - X_j)\widetilde{\Pi}_{l,0}(x_0, s) + \frac{k}{s}\chi(x_0 - X_l)\delta_{m,0} = 0. \tag{4.5b}$$

Taking the limit $t \to \infty$ in equations (4.4*a*) and (4.4*b*) generates the following equations for the total hitting probability $\pi_{l,m}(x_0)$:

$$0 = v\frac{\partial \pi_{l,1}(x_0)}{\partial x_0} - \beta[\pi_{l,1}(x_0) - \pi_{l,0}(x_0)], \tag{4.6a}$$

$$-k\chi(x_0 - X_l) = \alpha[\pi_{l,1}(x_0) - \pi_{l,0}(x_0)] - k\sum_{j=1,2}\chi(x_0 - X_j)\pi_{l,0}(x_0). \tag{4.6b}$$

First, solving equation (4.4*b*) for π_l^0 gives

$$\pi_{l,0}(x_0) = \frac{\alpha\pi_{l,1}(x_0) + k\chi(x_0 - X_l)}{\alpha + k\sum_{j=1,2}\chi(x_0 - X_j)}, \tag{4.7}$$

which can be substituted into equation (4.4*a*) to yield

$$\frac{\partial \pi_{l,1}(x_0)}{\partial x_0} = \frac{1}{v}\left(\beta - \frac{\alpha\beta}{\alpha + k\sum_{j=1,2}\chi(x_0 - X_j)}\right)\pi_{l,1}(x_0) - \frac{1}{v}\frac{\beta k\chi(x_0 - X_l)}{\alpha + k\sum_{j=1,2}\chi(x_0 - X_j)}. \tag{4.8}$$

Using the fact that $\pi_{1,1}(x_0) = 0$ for $x_0 > X_1 + a$ and $\pi_{2,1}1(x_0) = 0$ for $x_0 > X_2 + a$. We have

$$\pi_{l,1}(x_0) = A_l, \quad l = 1, 2, \tag{4.9a}$$

for $0 \leqslant x_0 \leqslant X_1 - a$,

$$\pi_{1,1}(x_0) = A_1 e^{\lambda(x_0 - X_1 + a)} - \left(e^{\lambda(x_0 - X_1 + a)} - 1\right), \quad \pi_{2,1}(x_0) = A_2 e^{\lambda(x_0 - X_1 + a)} \tag{4.9b}$$

for $X_1 - a < x_0 \leqslant X_1 + a$,

$$\pi_{1,1}(x_0) = 0, \quad \pi_{2,1}(x_0) = A_2 e^{2\lambda a}, \tag{4.9c}$$

for $X_1 + a < x_0 \leqslant X_2 - a$, and

$$\pi_{1,1}(x_0) = 0, \quad \pi_{2,1}(x_0) = A_2 e^{2\lambda a} e^{\lambda(x_0 - X_2 + a)} - \left(e^{\lambda(x_0 - X_2 + a)} - 1\right) \tag{4.9d}$$

for $X_2 - a < x_0 \leqslant X_2 + a$. We have used $f_0(0) = 0$, $f_k(0) = \lambda$ and imposed continuity conditions at $x = X_1 - a$, $x = X_1 + a$ and $x = X_2 - a$. Finally requiring $\pi_{1,1}(X_1 + a) = 0$ and $\pi_{2,1}(X_2 + a) = 0$ we determine the coefficients A_1, A_2, and thus obtain the expected hitting probabilities:

$$\Pi_1 = \pi_{1,1}(0) = A_1 = 1 - e^{-2a\lambda} = \Pi,$$

and

$$\Pi_2 = \pi_{2,1}(0) = A_2 = \left(1 - e^{-2\lambda a}\right)e^{-2\lambda a} = \Pi(1 - \Pi).$$

In order to calculate the conditional MFPT T_l to be absorbed by the lth target we need to solve equations (4.5*a*) and (4.5*b*). The latter gives

$$\widetilde{\Pi}_{l,0}(x_0, s) = \frac{\alpha\widetilde{\Pi}_{l,1}(x_0, s) + k\chi(x_0 - X_l)/s}{\alpha + s + k\sum_{j=1,2}\chi(x_0 - X_j)}, \tag{4.10}$$

which on substitution into equation (4.5*a*) yields

$$\frac{\partial\widetilde{\Pi}_{l,1}(x_0,s)}{\partial x_0} = \frac{1}{v}\left(\beta + s - \frac{\alpha\beta}{\alpha + s + k\sum_{j=1,2}\chi(x_0 - X_j)}\right)\widetilde{\Pi}_{l,1}(x_0,s) - \frac{1}{v}\left(\frac{\beta k\chi(x_0 - X_l)/s}{\alpha + s + k\sum_{j=1,2}\chi(x_0 - X_j)}\right). \tag{4.11}$$

Equation (4.11) can be solved along similar lines to (2.12), except now we have five domains: (1) $0 \leqslant x_0 \leqslant X_1 - a$, (2) $X_1 - a < x_0 \leqslant X_1 + a$, (3) $X_1 + a < x_0 \leqslant X_2 - a$, (4) $X_2 - a < x_0 \leqslant X_2 + a$ and (5) $x_0 > X_2 + a$. First note that $\Pi_{1,1}(x_0, t) = 0$ for $x_0 > X_1 + a$ and $\Pi_{2,1}(x_0, t) = 0$ for $x_0 > X_2 + a$. In the case of the first target ($l = 1$),

$$\widetilde{\Pi}_{1,1}(x_0,s) = a_1(s)\mathrm{e}^{f_0(s)[x_0 - X_1 + a]} \tag{4.12a}$$

for $0 \leqslant x_0 < X_1 - a$,

$$\widetilde{\Pi}_{1,1}(x_0,s) = b_1(s)\mathrm{e}^{f_k(s)[x_0 - X_1 + a]} + \frac{k}{s}\frac{h_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{f_k(s)[x_0 - X_1 + a]}\right) \tag{4.12b}$$

for $X_1 - a < x_0 \leqslant X_1 + a$, and $\widetilde{\Pi}_1^1(x_0, s) = 0$ for $x_0 \geqslant X_1 + a$. Here $h_k(s) = g_k(s) - v^{-1}$. Imposing continuity at $x = X_1 \pm a$ shows that

$$a_1(s) = b_1(s) = \frac{k}{s}\frac{h_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{-2af_k(s)}\right). \tag{4.13}$$

Since $kh_k(0)/f_k(0) = 1$, we see that

$$\Pi_1 = \lim_{s\to\infty} sa_1(s) = \Pi$$

as expected. Moreover the MFPT T_1 is

$$T_1 = \Pi^{-1}\int_0^{\infty}[\pi_{1,1}(0) - \Pi_{1,1}(0,t)]\mathrm{d}t = \Pi^{-1}\lim_{s\to 0}\left[\frac{\pi_1^1(0)}{s} - \widetilde{\Pi}_{1,1}(0,s)\right]. \tag{4.14}$$

It can be checked that this reproduces the MFPT (2.20) for a single target.

Turning to the second target ($l = 2$),

$$\widetilde{\Pi}_{2,1}(x_0,s) = a_2(s)\mathrm{e}^{f_0(s)[x_0 - X_1 + a]} \tag{4.15a}$$

for $0 \leqslant x_0 < X_1 - a$,

$$\widetilde{\Pi}_{2,1}(x_0,s) = a_2(s)\mathrm{e}^{f_k(s)[x_0 - X_1 + a]} \tag{4.15b}$$

for $X_1 - a < x_0 \leqslant X_1 + a$,

$$\widetilde{\Pi}_{2,1}(x_0,s) = b_2(s)\mathrm{e}^{f_0(s)[x_0 - X_2 + a]} \tag{4.15c}$$

for $X_1 + a < x_0 \leqslant X_2 - a$,

$$\widetilde{\Pi}_{2,1}(x_0,s) = b_2(s)\mathrm{e}^{f_k(s)[x_0 - X_2 + a]} + \frac{k}{s}\frac{h_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{f_k(s)[x_0 - X_2 + a]}\right) \tag{4.15d}$$

for $X_2 - a < x_0 \leqslant X_2 + a$,
and $\widetilde{\Pi}_{2,1}(x_0, s) = 0$ for $x_0 \geqslant X_2 + a$. Imposing continuity at $x = X_1 + a$ and $x = X_2 - a$ shows that

$$b_2(s) = \frac{k}{s}\frac{h_k(s)}{f_k(s)}\left(1 - \mathrm{e}^{-2af_k(s)}\right) \tag{4.16}$$

and

$$a_2(s) = \frac{k}{s}\frac{h_k(s)}{f_k(s)} \mathrm{e}^{-2af_k(s)} \mathrm{e}^{f_0(s)[X_1 - X_2 + 2a]} \left(1 - \mathrm{e}^{-2af_k(s)}\right). \tag{4.17}$$

Hence,

$$\Pi_2 = \lim_{s\to\infty} sa_2(s) = \Pi(1-\Pi)$$

as expected. The corresponding MFPT is

$$\begin{aligned} T_2 &= \frac{1}{\Pi(1-\Pi)} \int_0^\infty [\pi_{2,1}(0) - \Pi_{2,1}(0,t)]\mathrm{d}t = \frac{1}{\Pi(1-\Pi)} \lim_{s\to 0} \left[\frac{\pi_{2,1}(0)}{s} - \widetilde{\Pi}_{2,1}(0,s)\right] \\ &= 2a\mu_2 \left(1 - \frac{1}{\mathrm{e}^{2\lambda a} - 1}\right) + \mu_1(X_2 - 3a) + \mu_3. \end{aligned} \tag{4.18}$$

As seen in figure 9, the addition of a target between a searcher and its intended target qualitatively changes the behavior of the hitting probability as a function of the state transition rates. In particular, the hitting probability can now have a maximum value as a function of α. (A similar result holds on fixing α and varying β.) On the other hand, we find that the conditional MFPT is relatively insensitive to the presence of the upstream target, and still varies monotonically with α and β.

4.2. Pair of targets with resetting

Now suppose that we include resetting to the origin when in the moving state. We will develop the analysis using the probabilistic approach of section 3. Alternatively, one could use the renewal method and derive a backward equation for conditional survival probabilities along analogous lines to the analysis of a Brownian particle with resetting in an interval [31].

Consider the following set of FPTs:

$$\begin{aligned} \mathcal{T}_l &= \inf\{t > 0; X_l - a < X(t) \leqslant X_l + a, L(t) = l\}, \\ \mathcal{S} &= \inf\{t > 0; N_{\mathrm{reset}}(t) = 1, L(t) = 0\}, \\ \mathcal{R}_l &= \inf\{t > 0; X_k - a < X(\mathcal{S} + t) \leqslant X_k + a, L(\mathcal{S} + t) = l\}, \end{aligned}$$

for $l = 1, 2$. Recall that X_l is the position of the lth target and $L(t) = l > 0$ indicates that the particle has been absorbed by the lth target, whereas $L(t) = 0$ means that the particle has not been absorbed by any targets. Next we introduce the sets

$$\Omega_l = \{\mathcal{T}_l < \infty\}, \quad \Gamma_l = \{\mathcal{T}_0 < \mathcal{T}_l < \infty\} \subset \Omega_l,$$

where Ω_l is the set of all events for which the particle is eventually absorbed by the lth target, and Γ_l is the subset of events in Ω_l for which the particles resets at least once. Thus $\Omega_l \backslash \Gamma_l$ is the set of all events for which the particle is captured by the lth target cell without any resetting.

Let $\Pi_l^{(r)}$ denote the hitting probability that the particle finds the kth target in the presence of resetting. Then

$$\Pi_l^{(r)} := \mathbb{P}[\Omega_l] = \mathbb{P}[\Omega_l \backslash \Gamma_l] + \mathbb{P}[\Gamma_l]. \tag{4.19}$$

Next we have

$$\mathbb{P}[\Omega_l \backslash \Gamma_l] = \mathbb{P}[\mathcal{T}_l 1_{\Omega_l \backslash \Gamma_l} < \infty] = \int_0^\infty \mathrm{e}^{-r\tau} \frac{\mathrm{d}\Pi_{l,1}(0,\tau)}{\mathrm{d}\tau} \mathrm{d}\tau = r\widetilde{\Pi}_{l,1}(0,r).$$

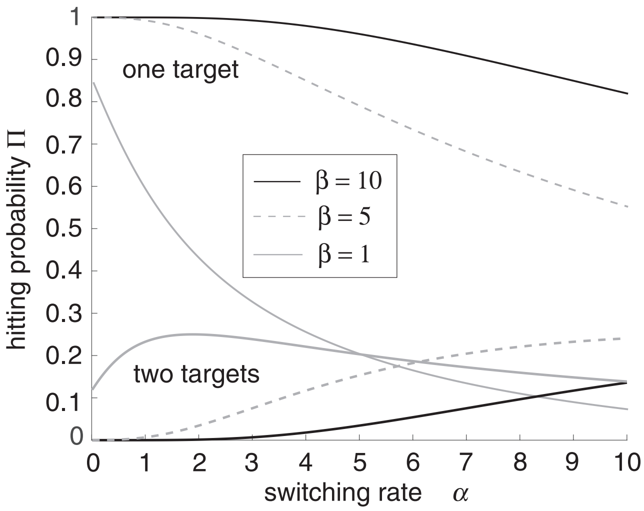

Figure 9. Anterograde unidirectional transport with two targets located at $X_1 = 5$ and $X_2 = 10$. The hitting probability Π_2 of the downstream target is plotted as a function of α for various β. Also shown are the corresponding results when the upstream target is absent. Other parameter values are $k = v = 1$ and $a = 1$.

We also have

$$\mathbb{P}[\Gamma_l] = \mathbb{P}[\mathcal{S} < \infty]\mathbb{P}[\mathcal{R}_l < \infty]$$

with

$$\mathbb{P}[\mathcal{S} < \infty] = \int_0^\infty r[1 - \Pi_{1,1}(0,\tau) - \Pi_{2,1}(0,\tau)]\mathrm{e}^{-r\tau}\mathrm{d}\tau = [1 - r\widetilde{\Pi}_{1,1}(0,r) - r\widetilde{\Pi}_{2,1}(0,r)].$$

Finally, from the strong Markov property $\mathbb{P}[\mathcal{R}_l < \infty] = \Pi_l^{(r)}$, so that

$$\Pi_l^{(r)} = r\widetilde{\Pi}_{l,1}(0,r) + [1 - r\widetilde{\Pi}_{1,1}(0,r) - r\widetilde{\Pi}_{2,1}(0,r)]\Pi_l^{(r)},$$

which gives

$$\Pi_l^{(r)} = \frac{\widetilde{\Pi}_{l,1}(0,r)}{\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)} = \frac{a_l(r)}{a_1(r) + a_2(r)}. \tag{4.20}$$

It immediately follows that $\Pi_1^{(r)} + \Pi_2^{(r)} = 1$, which means the particle is eventually absorbed by one of the targets. Example plots of $\Pi_1^{(r)}$ are shown in figures 10 and 11. The latter illustrates the fact that the hitting probabilities depend on the separation $X_1 - X_2$ of the two targets according to equation (4.17). As expected, the hitting probability of the upstream target is larger ($\Pi_1 > \Pi_2$). Moreover, for fixed r, decreasing the relative time spent in the ballistic (search) phase reduces (increases) Π_2.

Similarly, we can decompose the MFPT $T_l^{(r)} = \mathbb{E}[\mathcal{T}_l]$ as

$$T_l^{(r)} = \mathbb{E}[\mathcal{T}_l 1_{\Omega_l \setminus \Gamma_l}] + \mathbb{E}[\mathcal{T}_l 1_{\Gamma_l}], \tag{4.21}$$

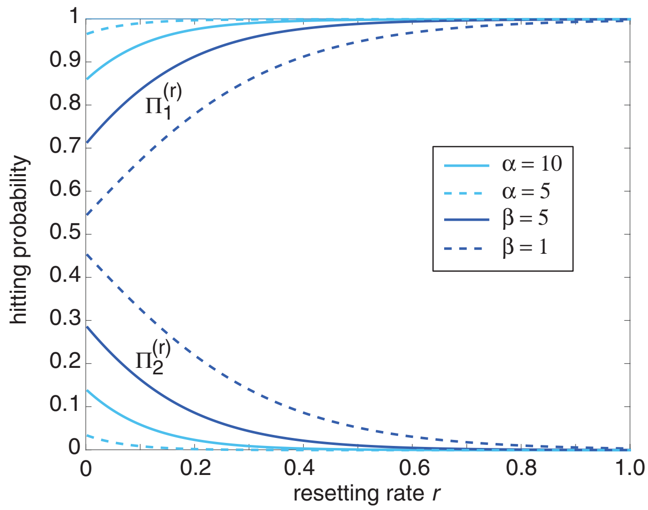

Figure 10. Hitting probabilities Π_l, $l = 1, 2$, for two competing targets and different switching rates plotted as a function of the resetting rate r. The hitting probability of the upstream target is larger ($\Pi_1 > \Pi_2$). Decreasing the relative time spent in the ballistic phase (smaller α) reduces Π_2, whereas decreasing the relative time spent in the search phase (smaller β) increases Π_2. The baseline rates are $\alpha = \beta = 10$. Other parameter values are $X_1 = 5$, $X_2 = 10$, $k = v = 1$ and $a = 1$.

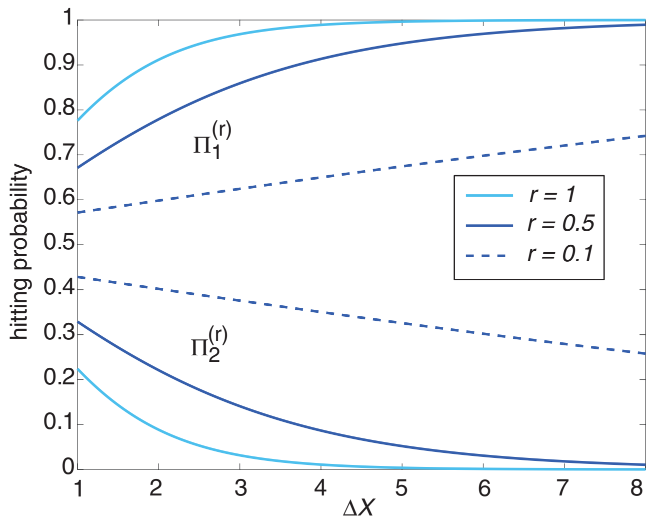

Figure 11. Hitting probabilities Π_l, $l = 1, 2$, for two competing targets and different switching rates plotting as a function of the target separation $\Delta X = X_2 - X_1$. Other parameter values are $\alpha = 10$, $\beta = 1$, $k = v = 1$ and $a = 1$.

with

$$\mathbb{E}[\mathcal{T}_l 1_{\Omega_l \setminus \Gamma_l}] = \int_0^\infty \tau \mathrm{e}^{-r\tau} \frac{\mathrm{d}\Pi_{l,1}(0,\tau)}{\mathrm{d}\tau} \mathrm{d}\tau = -\left[1 + r\frac{\mathrm{d}}{\mathrm{d}r}\right] \widetilde{\Pi}_{l,1}(0,r),$$

and

$$\mathbb{E}[\mathcal{T}_l 1_{\Gamma_l}] = \mathbb{E}[(\mathcal{S} + \mathcal{R}_l)1_{\Gamma_l}] = \mathbb{E}[\mathcal{S}1_{\Gamma_l}] + \mathbb{E}[\mathcal{R}_l 1_{\Gamma_l}]. \tag{4.22}$$

We now note that

$$\begin{aligned}\mathbb{E}[\mathcal{S}1_{\Gamma_l}] &= \mathbb{P}[\mathcal{R}_l < \infty] r \int_0^\infty \tau[1 - \Pi_{1,1}(0,\tau) - \Pi_{2,1}(0,\tau)]\mathrm{e}^{-r\tau}\mathrm{d}\tau \\ &= \left[\frac{1}{r} + r\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{\Pi}_{1,1}(0,r) + r\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{\Pi}_{2,1}(0,r)\right]\Pi_l^{(r)},\end{aligned}$$

and from the strong Markov property,

$$\mathbb{E}[\mathcal{R}_l 1_{\Gamma_l}] = T_l^{(r)}\mathbb{P}[\Gamma_l] = T_l^{(r)}[1 - r\widetilde{\Pi}_{1,1}(0,r) - r\widetilde{\Pi}_{2,1}(0,r)]\Pi_l^{(r)}.$$

Combining all of these results and using equation (4.20, we find

$$\begin{aligned} rT_l^{(r)}\widetilde{\Pi}_{l,1}(0,r) = \frac{\Pi_l^{(r)}}{r} &+ r\frac{\widetilde{\Pi}_{l,1}(0,r)}{\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)}\frac{\mathrm{d}}{\mathrm{d}r}\left[\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)\right] \\ &- r\frac{\mathrm{d}}{\mathrm{d}r}\widetilde{\Pi}_{l,1}(0,r) - \widetilde{\Pi}_{l,1}(0,r).\end{aligned}$$

Dividing through by $r\widetilde{\Pi}_{l,1}(0,r)$ then yields

$$T_l^{(r)} = \frac{1}{r}\left(\frac{1}{r[\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)]} - 1\right) - \frac{\mathrm{d}}{\mathrm{d}r}\ln\left[\frac{\widetilde{\Pi}_{l,1}(0,r)}{\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)}\right]. \tag{4.23}$$

Setting

$$\widetilde{Q}_1^{(r)}(0,r) = 1 - r[\widetilde{\Pi}_{1,1}(0,r) + \widetilde{\Pi}_{2,1}(0,r)], \tag{4.24}$$

which relates the total hitting probability with the survival probability of the particle, we obtain the final result

$$T_l^{(r)} = T^{(r)} - \frac{\mathrm{d}}{\mathrm{d}r}\ln \Pi_l^{(r)}, \tag{4.25}$$

with

$$T^{(r)} = \frac{\widetilde{Q}_1(0,r)}{1 - r\widetilde{Q}_1(0,r)} \tag{4.26}$$

the conditional MFPT that the particle is absorbed by either target. Note, in particular, that

$$\Pi_1^{(r)}T_1^{(r)} + \Pi_2^{(r)}T_2^{(r)} = (\Pi_1^{(r)} + \Pi_2^{(r)})T^{(r)} - \frac{\mathrm{d}}{\mathrm{d}r}\Pi_1^{(r)} - \frac{\mathrm{d}}{\mathrm{d}r}\Pi_1^{(r)} = T^{(r)},$$

since $\Pi_1^{(r)} + \Pi_2^{(r)} = 1$.

Equation (4.25) is identical in form to equation (38) of [31], which was derived for a Brownian particle in an interval using the renewal method. In the latter case, the particle has a unit probability of being absorbed at either end in the absence of resetting, whereas in the case of directed intermittent search there is a finite probability of no absorption without resetting. In figure 12 we show sample plots of the conditional MFPTs as a function of the resetting rate. Note that the presence of a downstream target actually lowers the optimal MFPT $T_1^{(r)}$ of the upstream target, since one is weighting sample paths that favor the first target. In the limit

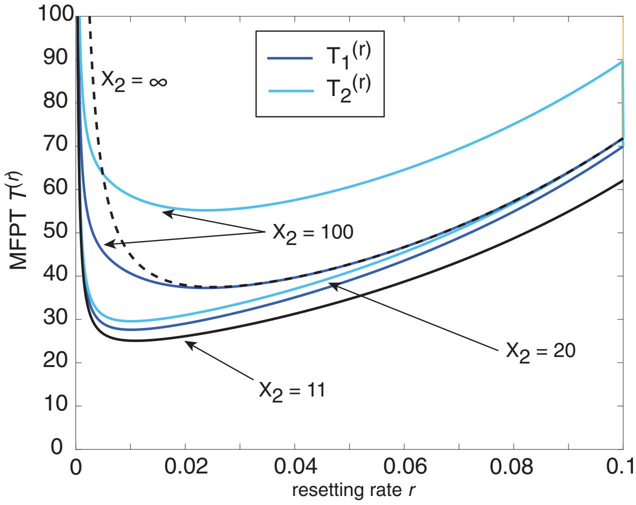

Figure 12. Conditional MFPTs $T_l^{(r)}$, $l = 1, 2$, as a function of the resetting rate for two competing targets and different positions X_2 of the downstream target. Other parameter values are $X_1 = 10$, $\alpha = \beta = 10$, $k = v = 1$ and $a = 1$. Dashed curve is MFPT for a single target at $X_1 = 10$. The two targets have approximately the same MFPT when $X_2 = 11$.

$X_2 \to \infty$, the presence of a second target has no effect and we recover the MFPT plot for a single target.

5. Discussion

In this paper we analyzed a biased velocity jump process in which a particle switches between a stationary state and right-moving ballistic state according to a two-state Markov process. The particle can detect (be absorbed by) a target at a fixed rate whenever it is in the stationary state and within range of the target. In addition, the particle can reset its position to the origin at some rate r. We determined hitting probabilities and (conditional) MFPTs as a function of various model parameters, both for a single target and a pair of competing targets. We also introduced an alternative method for analyzing stochastic processes with resetting, which is based on the use of conditional expectations, stopping times and the strong Markov property.

There are a number of possible extensions of the current work. First, one could consider a more complicated velocity jump process, in which the particle can exist in several distinct ballistic states, including motion in the retrograde direction (partially biased intermittent search). One example within the context of molecular motors is the so-called tug-of-war model of bidirectional intracellular transport, in which opposing groups of motors (e.g. kinesin and dynein) are attached to a vesicle [27, 29]. A second extension would be to consider higher-dimensional search processes such as motor-driven transport on cytoskeletal networks [8] or cytonemes searching for target cells in a hexagonal array, for example [13].

Finally, we note that there are certain analogies between the directed intermittent search process to multiple targets and recent studies of first-passage time processes with resetting, where there are two possible outcomes of the search process, namely, target detection or death [3, 14]. The latter builds upon the theory of so-called mortal walkers [35]. One major finding of these studies is that the splitting probability for success can be maximized as a function of the resetting rate, which we also found in our analysis, see figure 9. There are, however, some major differences. First, we consider unidirectional transport. Second, in the case of a single target, we have the analog of mortality in the absence of resetting, namely, the searcher passing beyond the target. However, 'mortality' disappears in the presence of resetting since the searcher eventually finds the target. In order to have multiple outcomes in the presence of resetting, we had to introduce more than one target.

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