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A Variational Method for Analyzing Stochastic Limit Cycle Oscillators*

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Abstract. We introduce a variational method for analyzing limit cycle oscillators in $\mathbb{R}^d$ driven by Gaussian noise. This allows us to derive exact stochastic differential equations for the amplitude and phase of the solution, which are accurate over times of order $(Cb\epsilon^{-1})$, where ϵ is the amplitude of the noise and b the magnitude of decay of transverse fluctuations. Within the variational framework, different choices of the amplitude-phase decomposition correspond to different choices of the inner product space $\mathbb{R}^d$. For concreteness, we take a weighted Euclidean norm, so that the minimization scheme determines the phase by projecting the full solution onto the limit cycle using Floquet vectors. Since there is coupling between the amplitude and phase equations, even in the weak noise limit, there is a small but nonzero probability of a rare event in which the stochastic trajectory makes a large excursion away from a neighborhood of the limit cycle. We use the amplitude and phase equations to bound the probability of it doing this: finding that the typical time the system takes to leave a neighborhood of the oscillator scales as $\exp(Cb\epsilon^{-1})$. We also show how the variational method provides a numerically tractable framework for calculating a stochastic phase, which we illustrate using a modified version of the Morris–Lecar model of a neuron.

Key words. stochastic oscillator, limit cycle, phase reduction

AMS subject classifications. 60H20, 60H25, 92C20, 92C15, 92C17

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1. Introduction.

A well-studied problem in dynamical systems theory is the construction and analysis of phase equations for stochastic limit cycle oscillators [8, 21, 2]. For example, consider the Ito stochastic differential equation (SDE) on $\mathbb{R}^d$,

$$du = F(u)dt + \sqrt{\epsilon}G(u)dW, \tag{1.1}$$

where $\epsilon > 0$ determines the noise strength and W_t is a vector of (correlated) Brownian motions with covariance $Q \in \mathbb{R}^{d\times d}$,

$$\mathbb{E}\left[W(t)W^\top(t)\right] = tQ.$$

Suppose that the deterministic equation for $\epsilon = 0$,

$$\frac{du}{dt} = F(u), \quad u \in \mathbb{R}^d, \tag{1.2}$$

with $F \in C^2$ has a stable periodic solution $u = U(t)$ with $U(t) = U(t+\Delta_0)$, where $\omega_0 = 2\pi/\Delta_0$ is the natural frequency of the oscillator. In state space the solution is an isolated attractive

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trajectory called a limit cycle. The dynamics on the limit cycle can be described by a uniformly rotating phase such that

$$\frac{d\theta}{dt} = \omega_0, \tag{1.3}$$

and $u = \Phi(\theta(t))$ with Φ a 2π-periodic function. Note that the phase is neutrally stable with respect to perturbations along the limit cycle—this reflects invariance of an autonomous dynamical system with respect to time shifts. Turning to the SDE (1.1), let us assume that the noise amplitude ϵ is sufficiently small given the rate of attraction to the limit cycle, so that deviations transverse to the limit cycle are also small (up to some exponentially large stopping time). This suggests that the definition of a phase variable persists in the stochastic setting, and one can derive a closed stochastic phase equation. Phase reduction typically proceeds in two steps [27, 13, 22, 31, 26]. First, one extends the definition of phase to a neighborhood of the limit cycle using the method of isochrons. However, as its stands, the isochronal phase depends on all of the state variables so that there is a coupling between the amplitude and phase dynamics. Therefore, the second step involves carrying out some form of perturbation expansion to obtain a closed equation for the phase dynamics, which is an approximation of the full dynamics. (The amplitude-phase decomposition can also be carried out using projection methods [14, 17, 4].)

An important issue is the timescale over which the reduced phase equation remains a good approximation of the full phase dynamics. In the case of SDEs, there are two distinct sources of error. The first is due to the fact that the solution of the reduced phase equation is an approximation of the exact isochronal phase, and is only accurate on timescales of $O(\epsilon^{-1})$. Of course, one could simply take the solution of the phase-reduced equation as the definition of the phase. This is particularly useful when considering the phase synchronization of coupled limit cycle oscillators, assuming convergence to a common reduced phase implies convergence to a common isochronal phase. On the other hand, if the occurrence of a particular event such as the initiation of a neuron's action potential is identified with a particular value of the isochronal phase, then errors arising from phase reduction could be significant. The second source of error, which is the main focus of our paper, arises from the observation that there is a nonzero probability that Gaussian fluctuations eventually lead to a large deviation of a stochastic trajectory from a neighborhood of the limit cycle. If the limit cycle has a finite basin of attraction then this could result in a noise-induced transition to another attractor. Moreover, even in the case of a globally attracting limit cycle, it is not clear that the notion of phase remains useful when a large deviation occurs. Since large deviations tend to happen on much longer timescales than the first type of error, it is necessary to construct a numerically computable phase variable that persists over these longer timescales. (One possibility would be to use the exact isochronal phase, but this can be difficult to compute.) Such a construction should allow a rigorous analysis of large deviations that yields bounds on the amplitude of transverse fluctuations about the limit cycle. (Such bounds also provide a necessary condition for the synchronization of coupled limit cycle oscillators.)

Therefore, in this paper, we introduce a variational method for carrying out the amplitude-phase decomposition of a stochastic limit cycle, in order to achieve the following goals: (i) to provide a numerically computable, exact definition of a phase that persists over exponentially long timescales; (ii) to obtain rigorous estimates for the expected time to escape a small neighborhood of the limit cycle. Within the variational framework, different choices of phase correspond to different choices of the inner product space $\mathbb{R}^d$. For concreteness, we take a weighted Euclidean norm, so that the minimization scheme determines the phase by projecting the full solution onto the limit cycle using Floquet vectors. Hence, in a neighborhood of the limit cycle the phase variable coincides with the isochronal phase [4]. This has the advantage that the amplitude and phase decouple to linear order in ϵ. We derive exact, implicit SDEs for the amplitude and phase, and use these to show that the expectation of the time it takes to leave an $O(\epsilon^\rho)$ neighborhood of the limit cycle, with $\rho < 1/2$, scales as $\exp(Cb\epsilon^{2\rho-1})$, for a constant C, where b is the magnitude of decay of the transverse fluctuations. These bounds are thus very useful in both the small noise limit, and the limit of strong decay of transverse fluctuations (as discussed in [26, 23]). Indeed they are accurate for finite ϵ/b and are more flexible and powerful than classical large deviations bounds. Our method is novel and uses a rescaling of time to demonstrate that the leading order behavior of the amplitude term is that of a stable Ornstein–Uhlenbeck process. These bounds also mean that the SDE for the phase is well-defined for times of order $\exp(Cb/\epsilon)$.

We note that a recent work [12] has obtained exponential bounds on the probability of the system leaving a neighborhood of the limit cycle that bear some similarities to ours. The primary goal of the cited paper is to understand the effect of noise on the winding number of a stochastic oscillator, in the limit as the magnitude of the noise tends to zero. The authors use an isochronal phase reduction to find that the expected time for the system to stay close to the limit cycle scales as $\exp(C\epsilon^{-1})$ for some constant C. We obtain a similar bound in the limit as $\epsilon \to 0$. However our bound is more general, and is much more accurate in the asymptotic regime of a strong decay towards the limit cycle, or a large period.

In the remainder of this section we briefly review phase reduction methods. The variational formulation is introduced in section 2, where we derive the exact amplitude and phase equations using Ito's lemma. In section 3 we carry out a perturbation expansion in the weak noise limit and compare the resulting phase equation with previous versions. We present a numerical example in section 4, based on a modified Morris–Lecar [20] conductance-based model of a neuron driven by extrinsic noise. We illustrate how the variational principle can be used as the basis for a numerical method to construct the phase, and compare the latter to the solution of the stochastic phase equation. Finally, exponential bounds on transverse fluctuations are derived in section 5.

1.1. Isochrons and phase–resetting curves. Suppose that we observe the unperturbed system (1.2) stroboscopically at time intervals of length Δ_0. This leads to a Poincaré mapping

$$u(t) \to u(t + \Delta_0) \equiv \mathcal{P}(u(t)).$$

This mapping has all points on the limit cycle as fixed points. Choose a point u^* on the cycle and consider all points in the vicinity of u^* that are attracted to it under the action of $\mathcal{P}$. They form a $(d-1)$-dimensional hypersurface $\mathcal{I}$, called an isochron, crossing the limit cycle at u^* (see Figure 1.1) [30, 18, 11, 6]. A unique isochron can be drawn through each point on the limit cycle (at least locally) so the isochrons can be parameterized by the phase, $\mathcal{I} = \mathcal{I}(\theta)$. Finally, the definition of phase is extended by taking all points $u \in \mathcal{I}(\theta)$ to have the same

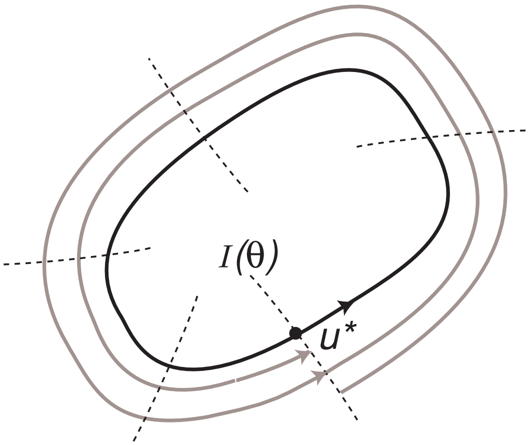

Figure 1.1. *Isochrons in the neighborhood of a stable limit cycle.*

phase, $\Theta(u) = \theta$, which then rotates at the natural frequency ω_0 (in the unperturbed case). Hence, for an unperturbed oscillator in the vicinity of the limit cycle we have

$$\omega_0 = \frac{d\Theta}{dt} = \sum_{k=1}^{d} \frac{\partial \Theta}{\partial u_k} \frac{du_k}{dt} = \sum_{k=1}^{d} \frac{\partial \Theta}{\partial u_k} F_k(u).$$

Now consider the deterministically perturbed system

$$\frac{du}{dt} = F(u) + \sqrt{\epsilon} G(u,t), \tag{1.4}$$

where, say, G is a Δ-periodic function of t. Keeping the definition of isochrons for the unperturbed system, one finds that to leading order

$$\frac{d\Theta}{dt} = \sum_{k=1}^{d} \frac{\partial \Theta}{\partial u_k} (F_k(u) + \sqrt{\epsilon} G_k(u,t)) = \omega_0 + \sqrt{\epsilon} \sum_{k=1}^{d} \frac{\partial \Theta}{\partial u_k} G_k(u,t).$$

As a further leading order approximation, deviations of u from the limit cycle are ignored. Hence, setting $u(t) = \Phi(\omega_0 t)$ with Φ the 2π-periodic solution on the limit cycle,

$$\frac{d\Theta}{dt} = \omega_0 + \sqrt{\epsilon} \sum_{k=1}^{d} \left. \frac{\partial \Theta}{\partial u_k} \right|_{u=\Phi} G_k(\Phi, t).$$

Finally, since points on the limit cycle are in 1:1 correspondence with the phase θ, one can set $U = U(\theta)$ and $\Theta(U(\theta)) = \theta$ to obtain the closed phase equation

$$\frac{d\theta}{dt} = \omega_0 + \sqrt{\epsilon} \sum_{k=1}^{d} R_k(\theta) G_k(\Phi(\theta), t), \tag{1.5}$$

where

$$R_k(\theta) = \left.\frac{\partial \Theta}{\partial u_k}\right|_{u=\Phi(\theta)} \tag{1.6}$$

is a 2π-periodic function of θ known as the kth component of the phase response curve (PRC).

It is well known that the PRC $R(\theta)$ can also be obtained as a 2π-periodic solution of the linear equation [7, 8, 21]

$$\omega_0 \frac{dR(\theta)}{d\theta} = -J(\theta)^\top \cdot R(\theta) \tag{1.7}$$

with the normalization condition

$$R(\theta) \cdot \frac{d\Phi(\theta)}{d\theta} = 1. \tag{1.8}$$

Here $J(\theta)^\top$ is the transpose of the Jacobian matrix $J(\theta)$, i.e.,

$$J_{jk}(\theta) \equiv \left.\frac{\partial F_j}{\partial u_k}\right|_{u=\Phi(\theta)}. \tag{1.9}$$

It should be noted that we can evaluate the multiplication of the Jacobian by the derivative of Φ by differentiating the unperturbed ODE on the limit cycle,

$$\omega_0 \frac{d\Phi}{d\theta} = F(\Phi(\theta)),$$

with respect to θ. This gives

$$\frac{d}{d\theta}\left(\frac{d\Phi}{d\theta}\right) = \omega_0^{-1} J(\theta) \cdot \frac{d\Phi}{d\theta}. \tag{1.10}$$

The next step is to assume that the above phase reduction procedure can also be applied to the SDE (1.1). This would then lead to the stochastic phase equation

$$d\theta = \omega_0 dt + \sqrt{\epsilon} \sum_{k,l=1}^{d} R_k(\theta) G_{kl}(\Phi(\theta)) dW_l(t). \tag{1.11}$$

However, this does not take proper account of stochastic calculus as expressed by Ito's lemma [10]. That is, the phase reduction procedure assumes that the ordinary rules of calculus apply. In the stochstic setting, this only holds if the multiplicative white noise term in (1.1) and (1.11) is interepreted in the sense of Stratonovich. However, the Ito form of the stochastic phase equation is more useful when calculating correlations, for example. Hence, converting (1.11) from Stratonovich to Ito using Ito's lemma gives [31, 26]

$$d\theta = \left[\omega_0 + \epsilon \sum_{k=1}^{d} Z_k'(\theta) Q_{kl} Z_l(\theta)\right] dt + \sqrt{\epsilon} \sum_{k}^{d} Z_k(\theta) dW_k(t), \tag{1.12}$$

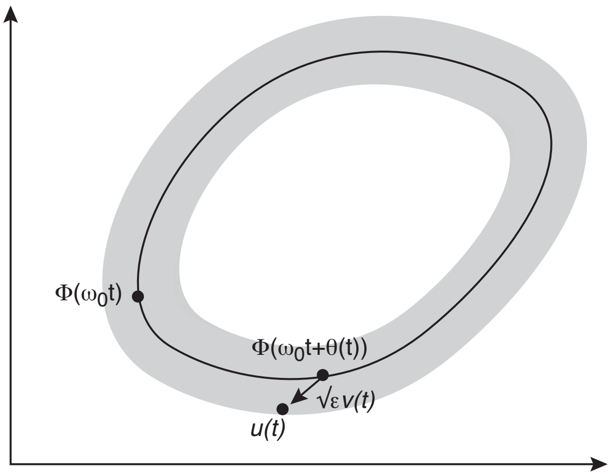

Figure 1.2. *Decomposition of the stochastic solution $u(t)$ into a random phase shift $\theta(t)$ along the deterministic limit cycle and a random transversal component $\sqrt{\epsilon}v(t)$.*

where we have set

$$Z_l(\theta) = \sum_{k=1}^{d} R_k(\theta) G_{kl}(\Phi(\theta)). \tag{1.13}$$

Hence, Ito's lemma yields an $O(\epsilon)$ contribution to the phase drift. Another subtle feature of the stochastic phase reduction procedure is that another $O(\epsilon)$ contribution occurs when taking into account perturbations transverse to the limit cycle [31]. However, the latter contribution is negligible if the limit cycle is strongly attracting [26].

1.2. Amplitude-phase decomposition. An alternative way to derive a stochastic phase equation is to explicitly decompose the solution of (1.1) into longitudinal (phase) and transverse (amplitude) fluctuations of the limit cycle [3, 17, 4]. The basic intuition is that Gaussian-like transverse fluctuations are distributed in a tube of radius $1/\sqrt{\epsilon}$ (up to some stopping time), whereas the phase around the limit cycle undergoes Brownian diffusion. Thus, the solution is decomposed in the form

$$u(t) = \Phi(\omega_0 t + \theta(t)) + \sqrt{\epsilon} v(t), \tag{1.14}$$

where the scalar random variable $\theta(t)$ represents the undamped random phase shift along the limit cycle, and $v(t)$ is a transversal perturbation; see Figure 1.2. Since there is no damping of fluctuations along the limit cycle, the random phase $\theta(t)$ is taken to undergo Brownian motion. However, it is important to note that the decomposition (1.14) is not unique, so that the precise definition of the phase depends on the particular method of analysis. For example, one study defines the phase so that there is no drift [17]. On the other hand, Gonze, Halloy, and Gaspard [14] focus on determining an effective phase diffusion coefficient based on a WKB approximation of solutions to the corresponding Fokker–Planck equation. Finally, Bonnin [4] combines an amplitude-phase decomposition with Floquet theory to show that if

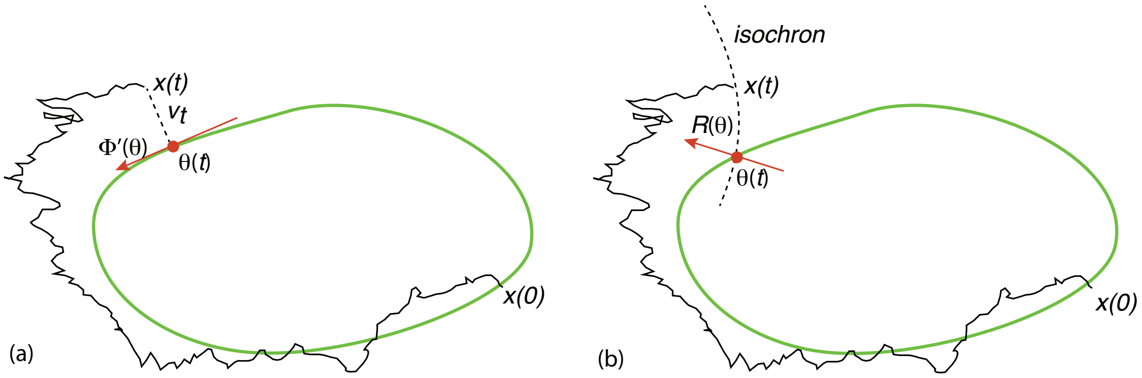

Figure 1.3. *The two different projection schemes highlighted in* [4]. (a) *Orthogonal projection with respect to the Euclidean norm of the solution $x(t)$ at time t onto the limit cycle. Response to perturbations depends on the tangent vector to the limit cycle, $\Phi'(\theta)$.* (b) *The method of isochrons determines the phase $\theta(t)$ by tracing where the isochron through $x(t)$ intersects the limit cycle.The response to perturbations depends on the PRC $R(\theta)$, which is normal to the isochron at the point of intersection with the limit cycle.*

Floquet vectors are used, then the resulting phase variable in a neighborhood of the limit cycle coincides with the asymptotic phase based on isochrons; see Figure 1.3.

2. Variational method.

Suppose that the deterministic ODE

$$\frac{du_t}{dt} = F(u_t), \quad u_t \in \mathbb{R}^d, \tag{2.1}$$

supports a stable periodic solution of the form $u_t = \Phi(\omega_0 t)$ with $\Phi(\omega_0 t + 2\pi n) = \Phi(t)$ for all integers n, and $\Delta_0 = 2\pi/\omega_0$ is the fundamental period of the oscillator. We are interested in deriving a stochastic equation for the effective phase of the oscillator when the system is perturbed by weak noise. Therefore, consider the Ito SDE[1]

$$du_t = F(u_t)dt + \sqrt{\epsilon}G(u_t)dW_t, \tag{2.2}$$

where $\epsilon > 0$ determines the noise strength. Here W_t is a vector of (potentially correlated) Brownian motions with covariance $Q \in \mathbb{R}^{d\times d}$,

$$\mathbb{E}\left[W_t W_t^\top\right] = tQ.$$

In the above, G is a Lipschitz map from $\mathbb{R}^d \to \mathbb{R}^{d\times d}$. (Note that the vector of Brownian motions need not have the same dimension as u_t.) Throughout this paper, for any matrix A, $\|A\|$ denotes the spectral norm. We assume a uniform bound on the spectral norm of G, i.e., there exists a constant λ_G such that

$$\sup_{u\in\mathbb{R}^d} \|G(u)\| \leq \lambda_G. \tag{2.3}$$

[1]It would be straightforward to extend the results of the paper if we were to interpret the stochastic integrals in the Stratonovich sense.

In the presence of noise we wish to decompose the solution u_t into two components: the "closest" point of $\Phi(\beta_t)$ to u_t for a *phase* β_t, and an "error" v_t that represents the amplitude of transversal fluctuations:

$$u_t = \Phi(\beta_t) + \sqrt{\epsilon} v_t, \quad v_t \in \mathbb{R}^d. \tag{2.4}$$

However, as pointed out in section 1.2, such a decomposition is not unique unless we impose an additional mathematical constraint. We will adapt a variational principle previously introduced by Inglis and Maclaurin [16] within the context of traveling waves in stochastic neural fields; see also [19, 15]. First, we must introduce a little Floquet theory.

2.1. Floquet decomposition and weighted norm. For any $0 \leq t$, define $\Pi(t) \in \mathbb{R}^{d\times d}$ to be the following fundamental matrix for the ODE

$$\frac{dz}{dt} = J(t)z, \tag{2.5}$$

where $J(t)$ is the Jacobian of F evaluated at $x = \Phi(\omega_0 t)$. That is,

$$\Pi(t) := \big(z_1(t)|z_2(t)|\dots|z_d(t)\big),$$

where $z_i(t)$ satisfies (2.5), $z_1(0) = \Phi'(0)$, and $\{z_i(0)\}_{i=1}^d$ is an orthogonal basis for $\mathbb{R}^d$. Floquet theory states that there exists a diagonal matrix $\mathcal{S} = \mathrm{diag}(\nu_1, \dots, \nu_d)$ whose diagonal entries are the Floquet characteristic exponents, such that

$$\Pi(t) = P\big(\omega_0 t\big)\exp\big(t\mathcal{S}\big)P^{-1}(0) \tag{2.6}$$

with $P(\theta)$ a 2π-periodic matrix whose first column is $\Phi'(\omega_0 t)$ and $\nu_1 = 0$. That is, $P^{-1}(\theta)\Phi'(\theta)$ $= \mathbf{e}$ with $\mathbf{e}_j = \delta_{1,j}$. In order to simplify the following notation, we will assume throughout this paper that the Floquet multipliers are real and hence $P(\theta)$ is a real matrix. One could readily generalize these results to the case that $\mathcal{S}$ is complex. The limit cycle is taken to be stable, meaning that for a constant $b > 0$, for all $2 \leq i \leq d$,

$$\nu_i \leq -b. \tag{2.7}$$

Since $F \in \mathcal{C}^2$, it follows that $P \in \mathcal{C}^2$. Furthermore $P^{-1}(\theta)$ exists for all θ, since $\Pi^{-1}(t)$ exists for all t.

The above Floquet decomposition motivates the following weighted inner product: for any $\theta \in \mathbb{R}$, denoting the standard Euclidean dot product on $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$,

$$\langle u, v\rangle_\theta = \big\langle P^{-1}(\theta)u, P^{-1}(\theta)v\big\rangle$$

and $\|u\|_\theta = \sqrt{\langle u, u\rangle_\theta}$. This weighting is useful for two reasons: it leads to a leading order separation of the phase from the amplitude (see section 3) and it facilitates the strong bounds of section 4 because the weighted amplitude always decays, no matter what the phase is. The former is a consequence of the fact that the matrix $P^{-1}(\theta)$ generates a coordination transformation in which the phase in a neighborhood of the limit cycle coincides with the asymptotic phase defined using isochrons (see also [4]). This is reflected by the following relationship between the tangent vector to the limit cycle, $\Phi'(\theta)$, and the PRC $R(\theta)$ of (1.6):

$$\mathfrak{M}(\theta)P^\top(\theta)R(\theta) = P^{-1}(\theta)\Phi'(\theta), \tag{2.8}$$

where

$$\mathfrak{M}(\theta) := \left\|P^{-1}(\theta)\Phi'(\theta)\right\|^2. \tag{2.9}$$

We will proceed by defining $R(\theta)$ according to (2.8) and showing that it satisfies the adjoint equation (1.7). We will need the relation

$$\omega_0 P'(\theta) = J(\theta)P(\theta) - P(\theta)\mathcal{S}, \tag{2.10}$$

which can be obtained by differentiating (2.6). Differentiating both sides of (2.8) with respect to θ, we have

$$\mathfrak{M}'P^\top R + \mathfrak{M}P^T R' + \mathfrak{M}(P^\top)'R = P^{-1}\Phi'' + (P^{-1})'\Phi' \tag{2.11}$$

with

$$\mathfrak{M}' = 2\left\langle P^{-1}\Phi'' + (P^{-1})'\Phi', P^{-1}\Phi'\right\rangle.$$

Equation (2.10) implies that

$$\omega_0(P^\top(\theta))' = P^\top(\theta)J^\top(\theta) - \mathcal{S}P^\top(\theta)$$

and

$$\omega_0(P^{-1}(\theta))' = -P^{-1}(\theta)J(\theta) + \mathcal{S}P^{-1}(\theta).$$

We have used the fact that $\mathcal{S}$ is a diagonal matrix and $P^{-1}P' + (P^{-1})'P = 0$ for any square matrix. Substituting these identities in (2.11) yields

$$\begin{aligned}\mathfrak{M}'P^\top R + \mathfrak{M}P^T(R' + \omega_0^{-1}J^\top R) - \omega_0^{-1}\mathfrak{M}\mathcal{S}P^\top R\\ = P^{-1}[\Phi'' - \omega_0^{-1}J\Phi'] + \omega_0^{-1}\mathcal{S}P^{-1}\Phi'\end{aligned}$$

and

$$\mathfrak{M}' = \left\langle P^{-1}[\Phi'' - \omega_0^{-1}J\Phi'] + \omega_0^{-1}\mathcal{S}P^{-1}\Phi', P^{-1}\Phi'\right\rangle.$$

Now note that Φ' satisfies (1.10) and $\mathcal{S}P^{-1}\Phi' = 0$. The latter follows from the condition $P(\theta)^{-1}\Phi'(\theta) = \mathbf{e}$ and $\mathcal{S}\mathbf{e} = \nu_1 = 0$. It also holds that $\mathfrak{M}'(\theta) = 0$. (In fact, for the specific choice of $P(\theta)$, we have $\mathfrak{M}(\theta) = \langle \mathbf{e}, \mathbf{e}\rangle = 1$.) Finally, from the definition of ($R(\theta)$, (2.8)), we deduce that $\mathcal{S}P^\top(\theta)R(\theta) = 0$ and hence

$$\mathfrak{M}P^T(R' + \omega_0^{-1}J^\top R) = 0.$$

Since $P^T(\theta)$ is nonsingular for all θ, R satisfies (1.7) and can thus be identified as the PRC.

2.2. Defining the stochastic phase using a variational principle. We can now state the variational principle for the stochastic phase: β_t is determined by requiring $\beta_t = a_t(\theta_t)$, where $a_t(\theta_t)$ for a prescribed time dependent weight θ_t is a local minimum of the following variational problem,

$$\inf_{a \in \mathcal{N}(a(\theta_t))} \|u_t - \Phi(a)\|_{\theta_t} = \|u_t - \Phi(a_t(\theta_t))\|_{\theta_t} \tag{2.12}$$

with $\mathcal{N}(a_t(\theta_t))$ denoting a sufficiently small neighborhood of $a_t(\theta_t)$. The minimization scheme is based on the orthogonal projection of the solution onto the limit cycle with respect to the weighted Euclidean norm at some θ_t. We will derive an exact SDE for β_t (up to some stopping time) by considering the first derivative

$$\mathcal{G}_0(z, a, \theta) := \frac{\partial}{\partial a}\|z - \Phi(a)\|_\theta^2 = -2\left\langle z - \Phi(a), \Phi'(a)\right\rangle_\theta . \tag{2.13}$$

At the local minimum,

$$\mathcal{G}_0(u_t, \beta_t, \theta_t) = 0. \tag{2.14}$$

We stipulate that the location of the weight must coincide with the location of the minimum, i.e., $\beta_t = \theta_t$, so that β_t must satisfy the implicit equation

$$\mathcal{G}(u_t, \beta_t) := \mathcal{G}_0(u_t, \beta_t, \beta_t) = 0. \tag{2.15}$$

It will be seen that, up to a stopping time τ, there exists a unique continuous solution to the above equation. Note that we could have defined β_t according to

$$\inf_{a \in \mathcal{N}(\beta_t)} \|u_t - \Phi(a)\|_a = \|u_t - \Phi(\beta_t)\|_{\beta_t}, \tag{2.16}$$

which might seem more intuitive. However to leading order in $(u_t - \Phi(\beta_t))$, the above two schemes are equivalent, and we prefer the former because it leads to simpler equations.

Define $\mathcal{H}(z, a) \in \mathbb{R}$ according to

$$\begin{aligned} \mathcal{H}(z, a) &:= \frac{1}{2}\frac{\partial \mathcal{G}(z, a)}{\partial a} = \frac{1}{2}\left.\frac{\partial \mathcal{G}_0(z, a, \theta)}{\partial a}\right|_{\theta = a} + \frac{1}{2}\left.\frac{\partial \mathcal{G}_0(z, a, \theta)}{\partial \theta}\right|_{\theta = a} \\ &= 1 - \left\langle z - \Phi(a), \Phi''(a)\right\rangle_a - \left\langle z - \Phi(a), \frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\Phi'(a)\right\rangle, \end{aligned} \tag{2.17}$$

where we have used the fact that $\|\Phi'(a)\|_a^2 = 1$, which we proved in the previous section. Assume that initially $\mathcal{H}(u_0, \beta_0) > 0$. We then seek an SDE for β_t that holds for all times less than the stopping time τ,

$$\tau = \inf\{s \geq 0 : \mathcal{H}(u_s, \beta_s) = 0\}. \tag{2.18}$$

The implicit function theorem guarantees that a unique continuous β_t exists until this time.

It is a consequence of Theorem 5.1 in section 5 that there exist constants $C, \widetilde{C} > 0$ such that

$$\mathbb{P}\Big(\tau \le \exp\left(Cb\epsilon^{-1}\right)\Big) \le \exp\left(-\widetilde{C}b\epsilon^{-1}\right),$$

where we recall that b is the lower bound on the rate of decay of the Floquet exponents.

In order to derive the SDE for β_t, we apply Ito's lemma to the identity

$$d\mathcal{G}_t := d\mathcal{G}(u_t, \beta_t) = 0 \tag{2.19}$$

with du_t given by (2.2) and $d\beta_t$ taken to satisfy an SDE of the form

$$d\beta_t = V(u_t, \beta_t)dt + \sqrt{\epsilon}\,\langle B(u_t, \beta_t), G(u_t)dW_t\rangle_{\beta_t} \tag{2.20}$$

for functions V and B that we determine below. Using the definition of $\mathcal{G}(u_t, \beta_t, \beta_t)$, $d\mathcal{G}_t$ is found to be

$$\begin{aligned} d\mathcal{G}_t = &- 2\left\langle du_t, \Phi'(\beta_t)\right\rangle_{\beta_t} + \left.\frac{\partial \mathcal{G}_t}{\partial a}\right|_{a=\beta_t} d\beta_t + \frac{1}{2}\left.\frac{\partial^2 \mathcal{G}_t}{\partial a^2}\right|_{a=\beta_t} d\beta_t d\beta_t - 2\left\langle du_t, \Phi''(\beta_t)d\beta_t\right\rangle_{\beta_t} \\ &- 2\left\langle du_t, \frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\Big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle d\beta_t. \end{aligned} \tag{2.21}$$

Note that we only include the dt contributions from the quadratic differential terms involving the products $du_t d\beta_t$ and $d\beta_t d\beta_t$, which are also known as cross variations. In particular, writing $K(u_t, \beta_t) = G^\top(u_t)[P(\beta_t)P^\top(\beta_t)]^{-1}$,

$$d\beta_t d\beta_t = \epsilon\,\langle K(u_t, \beta_t)B(u_t, \beta_t), QK(u_t, \beta_t)B(u_t, \beta_t)\rangle\, dt, \tag{2.22}$$

$$\begin{aligned} \left\langle du_t, \Phi''(\beta_t)d\beta_t\right\rangle_{\beta_t} &= \sqrt{\epsilon}\,\left\langle G(u_t)dW_t, \Phi''(\beta_t)d\beta_t\right\rangle_{\beta_t} \\ &= \epsilon\,\left\langle G(u_t)dW_t, \Phi''(\beta_t)\langle B(u_t, \beta_t), G(u_t)dW_t\rangle_{\beta_t}\right\rangle_{\beta_t} \\ &= \epsilon\,\left\langle K(u_t, \beta_t)\Phi''(\beta_t), QK(u_t, \beta_t)B(u_t, \beta_t)\right\rangle dt \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\left\langle du_t, \frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\Big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle d\beta_t \\ &\qquad = \epsilon\left\langle G^\top(u_t)\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\Big|_{a=\beta_t}\Phi'(\beta_t), QK(u_t, \beta_t)B(u_t, \beta_t)\right\rangle. \end{aligned} \tag{2.24}$$

Substituting (2.20), (2.22), and (2.23) into (2.21) yields an SDE of the form

$$d\mathcal{G}_t = \mathcal{V}(u_t, \beta_t)dt + \sqrt{\epsilon}\langle \mathcal{B}(u_t, \beta_t), G(u_t)dW_t\rangle_{\beta_t}. \tag{2.25}$$

In order that (2.19) be satisfied, we require that both terms on the right-hand side of the above equation are zero, which will determine V and B. First, we have

$$\begin{aligned} 0 := \frac{1}{2}\,\langle \mathcal{B}(u_t, \beta_t), G(u_t)dW_t\rangle_{\beta_t} = \mathcal{H}(u_t, \beta_t)\,\langle B(u_t, \beta_t), G(u_t)dW_t\rangle_{\beta_t} \\ - \left\langle G(u_t)dW_t, \Phi'(\beta_t)\right\rangle_{\beta_t}. \end{aligned}$$

Since for all times less than τ, $\mathcal{H}(u_t,\beta_t)>0$, it follows that $\mathcal{H}^{-1}$ exists and, hence,

$$B(u_t,\beta_t)=\mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t). \tag{2.26}$$

Second,

$$0:=\mathcal{V}(u_t,\beta_t)dt=\left[\frac{\partial \mathcal{G}_t}{\partial a}\bigg|_{a=\beta_t}V(u_t,\beta_t)-2\left(\left\langle F(u_t),\Phi'(\beta_t)\right\rangle_{\beta_t}dt+\epsilon\kappa(u_t,\beta_t)\right)\right]dt \tag{2.27}$$

with

$$\begin{aligned}\epsilon\kappa(u_t,\beta_t)dt:=&-\frac{1}{4}\frac{\partial^2\mathcal{G}_t}{\partial a^2}\bigg|_{a=\beta_t}d\beta_t d\beta_t\quad+\left\langle du_t,\Phi''(\beta_t)d\beta_t\right\rangle_{\beta_t}\\&+\left\langle du_t,\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle d\beta_t.\end{aligned} \tag{2.28}$$

The cross variations (2.22) and (2.23) can now be evaluated using (2.26):

$$d\beta_t d\beta_t=\epsilon\mathcal{H}(u_t,\beta_t)^{-2}\left\langle K(u_t,\beta_t)\Phi'(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle dt, \tag{2.29}$$

$$\left\langle du_t,\Phi''(\beta_t)d\beta_t\right\rangle_{\beta_t}=\epsilon\mathcal{H}(u_t,\beta_t)^{-1}\left\langle K(u_t,\beta_t)\Phi''(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle dt, \tag{2.30}$$

and

$$\begin{aligned}&\left\langle du_t,\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle d\beta_t\\&\quad=\epsilon\mathcal{H}(u_t,\beta_t)^{-1}\left\langle G^\top(u_t)\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle dt.\end{aligned} \tag{2.31}$$

Equations (2.27)–(2.30) determine the drift term V so that

$$d\beta_t=\mathcal{H}(u_t,\beta_t)^{-1}\left[\left(\left\langle F(u_t),\Phi'(\beta_t)\right\rangle_{\beta_t}+\epsilon\kappa(u_t,\beta_t)\right)dt+\sqrt{\epsilon}\left\langle G(u_t)dW_t,\Phi'(\beta_t)\right\rangle_{\beta_t}\right], \tag{2.32}$$

where

$$\begin{aligned}\kappa(u_t,\beta_t):=&\,\mathcal{H}(u_t,\beta_t)^{-1}\left\langle K(u_t,\beta_t)\Phi''(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle\\&+\mathcal{H}(u_t,\beta_t)^{-1}\left\langle G^\top(u_t)\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle\\&+\frac{\mathcal{H}(u_t,\beta_t)^{-2}}{2}\Bigg[\left\langle u_t-\Phi(\beta_t),\Phi'''(\beta_t)\right\rangle_{\beta_t}-\left\langle\Phi'(\beta_t),\Phi''(\beta_t)\right\rangle_{\beta_t}\\&+\left\langle u_t-\Phi(\beta_t),\frac{d^2}{da^2}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle\\&+2\left\langle u_t-\Phi(\beta_t),\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi''(\beta_t)\right\rangle\\&-\left\langle\Phi'(\beta_t),\frac{d}{da}\left[P(a)P^\top(a)\right]^{-1}\big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle\Bigg]\left\langle K(u_t,\beta_t)\Phi'(\beta_t),QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle.\end{aligned} \tag{2.33}$$

Finally, recall that the amplitude term v_t satisfies

$$\sqrt{\epsilon} v_t = u_t - \Phi_{\beta_t}. \tag{2.34}$$

Hence, applying Ito's lemma

$$\begin{aligned}
\sqrt{\epsilon} dv_t &= du_t - \Phi'(\beta_t) d\beta_t - \frac{1}{2}\Phi''(\beta_t) d\beta_t d\beta_t \\
&= \Big[F(u_t) - \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)\left(\big\langle F(u_t), \Phi'(\beta_t)\big\rangle_{\beta_t} + \epsilon\kappa(u_t,\beta_t)\right) \\
&\qquad - \frac{\epsilon}{2}\Phi''(\beta_t)\mathcal{H}(u_t,\beta_t)^{-2}\left\langle K(u_t,\beta_t)\Phi'(\beta_t), QK(u_t,\beta_t)\Phi'(\beta_t)\right\rangle\Big] dt \\
&\qquad + \sqrt{\epsilon}\left[G(u_t)dW_t - \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)\big\langle G(u_t)dW_t, \Phi'(\beta_t)\big\rangle_{\beta_t}\right],
\end{aligned} \tag{2.35}$$

where we have used (2.2), and the differentials $d\Phi_t = F(\Phi_t)dt$ and $d\Phi_{\beta_t} = \Phi' d\beta_t + \frac{1}{2}\Phi'' d\beta_t d\beta_t$.

3. Weak noise limit.

In order to obtain a closed equation for β_t we carry out a perturbation analysis in the weak noise limit, and compare the variational phase equation with various versions of the phase equations previously derived using isochronal phase reduction methods; see section 1.1. We demonstrate that the linearization of our phase equation is accurate over timescales of order ϵ^{-1}. This means that the timescale over which the linearization of our phase equation is accurate is of the same order as the isochronal phase equation. It should be noted that, as we explain in more detail in section 5, our method possesses the additional virtue of having an analytic SDE that is accurate over timescales of order $O(\exp(Cb\epsilon^{-1}))$, where b is the rate of decay of transverse fluctuations.

Suppose that $0 < \epsilon \ll 1$ and set $u_t = \Phi(\beta_t)$ on the right-hand side of (2.32). That is, we drop any v_t-dependent terms. Setting $\beta_t = \theta$, we obtain the explicit stochastic phase equation

$$d\theta = [\omega_0 + \epsilon\widehat{\kappa}(\theta)]dt + \sqrt{\epsilon}\Big\langle G(\Phi(\theta))dW_t, R(\theta)\Big\rangle \tag{3.1}$$

with $R(\theta)$ identified as the normal to the isochron crossing the limit cycle at θ (see Figure 1.3(b) and (2.8)):

$$R(\theta) = [PP^\top(\theta)]^{-1}\Phi'(\theta), \tag{3.2}$$

since $\mathfrak{M}(\theta) = 1$. We have used the identity

$$\big\langle \Phi'(\theta)\big), R(\theta)\big\rangle = 1 \tag{3.3}$$

and $F(\Phi(\theta)) = \omega_0\Phi'(\theta)$. Equation (3.1) has a similar form to the isochronal phase equation (1.12). However, in contrast to the latter, there is no $O(\epsilon)$ contribution to the drift of the form

$\langle Z'(\theta), QZ(\theta)\rangle$ since we take the noise in SDE (2.2) to be Ito rather than Stratonovich. Thus, the $O(\epsilon)$ drift term $\widehat{\kappa}(\theta)$ in (3.1) is the analog of the contributions from transverse fluctuations identified in [31, 26].

As highlighted by Bonnin [4], although neglecting the coupling between the phase and amplitude dynamics by setting $v_t = 0$ yields a closed equation for the phase, it does lead to imprecision at short and intermediate times. (Errors at longer times due to large deviations from the limit cycle will be addressed in section 4.) Here we show that taking into account the amplitude coupling only results in $O(\epsilon)$ contributions to the drift, not $O(\sqrt{\epsilon})$. Neglecting v_t-independent $O(\epsilon)$ drift terms, (2.32) becomes

$$d\beta_t = \Big\langle F(u_t), \mathfrak{R}(u_t,\beta_t)\Big\rangle_{\beta_t} dt + \sqrt{\epsilon}\Big\langle G(u_t)dW_t, \mathfrak{R}(u_t,\beta_t)\Big\rangle_{\beta_t}, \tag{3.4}$$

where

$$\mathfrak{R}(u_t,\beta_t) = \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t). \tag{3.5}$$

Suppose that we rewrite $\mathfrak{R}$ as a function $\widehat{\mathfrak{R}}$ of β_t and v_t using (2.17): $\mathfrak{R}(u_t,\beta_t) = \widehat{\mathfrak{R}}(v_t,\beta_t)$ with

$$\widehat{\mathfrak{R}}(v_t,\beta_t) = \left(1 - \sqrt{\epsilon}\Big\langle v_t, \Phi''(\beta_t)\Big\rangle_{\beta_t} - \sqrt{\epsilon}\left\langle v_t, \frac{d}{da}\big[P(a)P^\top(a)^{-1}\big]\big|_{a=\beta_t}\Phi'(\beta_t)\right\rangle\right)^{-1}\Phi'(\beta_t).$$

Let us define

$$H(v,\theta) = \Big\langle F(\Phi(\theta)+\sqrt{\epsilon}v), \widehat{\mathfrak{R}}(v,\theta)\Big\rangle_\theta. \tag{3.6}$$

In the phase equation (3.1) we set $v = 0$ and used $H(0,\theta) = \omega_0$. Suppose that we now include higher-order terms by Taylor expanding $H(v,\theta)$ with respect to v. In particular, consider the first derivative

$$\begin{aligned}\frac{\partial H}{\partial v}(0,\theta)\cdot v &= \sqrt{\epsilon}\,\mathfrak{M}^{-1}\Big\langle J(\theta)\cdot v, \Phi'(\theta)\Big\rangle_\theta \\ &\quad \sqrt{\epsilon}\,\mathfrak{M}^{-2}\Big\langle F\big(\Phi(\theta)\big), \Phi'(\theta)\Big\rangle_\theta\left[\Big\langle v, \Phi''(\theta)\Big\rangle_\theta + \left\langle v, \frac{d}{da}\big[P(a)P^\top(a)^{-1}\big]\Big|_{a=\theta}\Phi'(\theta)\right\rangle\right] \\ &= \sqrt{\epsilon}\left\langle J(\theta)\cdot v, \Phi'(\theta)\right\rangle_\theta + \sqrt{\epsilon}\,\omega_0\frac{d}{d\theta}\Big\langle v, \Phi'(\theta)\Big\rangle_\theta,\end{aligned}$$

since $\mathfrak{M}(\theta) = 1$ and $\Big\langle F\big(\Phi(\theta)\big), \Phi'(\theta)\Big\rangle_\theta = \omega_0$. Observe that

$$\begin{aligned}
\Big\langle J(\theta)\cdot v, \Phi'(\theta)\Big\rangle_\theta &= \Big\langle P^{-1}(\theta)J(\theta)\cdot v, P^{-1}(\theta)\Phi'(\theta)\Big\rangle \\
&= \Big\langle J(\theta)\cdot v, [P(\theta)P^\top(\theta)]^{-1}\Phi'(\theta)\Big\rangle \\
&= \Big\langle v, J(\theta)^\top \cdot R(\theta)\Big\rangle \\
&= -\omega_0\Big\langle v, R'(\theta)\Big\rangle \\
&= -\omega_0\Big\langle v, \frac{d}{d\theta}\Big\{\big[P(\theta)P^\top(\theta)\big]^{-1}\Phi'(\theta)\Big\}\Big\rangle \\
&= -\omega_0\frac{d}{d\theta}\Big\langle v, \Phi'(\theta)\Big\rangle_\theta,
\end{aligned}$$

where in the third last line we have used (1.7), and in the second last line we have used (3.2).

We have thus proven that the phase equation decouples from the amplitude equation at $O(\sqrt{\epsilon})$, which is consistent with the analysis of [4]. Since the errors in the SDE are of $O(\epsilon)$, this linearization of our phase equation is accurate over timescales of order $O(\epsilon^{-1})$, which is the same order as the isochronal phase equation.

4. Example: Stochastic Morris–Lecar model.

So far we have presented a new variational method for determining the phase of a limit cycle oscillator driven by multiplicative white noise. One of the motivations for this formulation is that it allows us to derive exponential bounds on transverse fluctuations, as detailed in section 5. However, it is also important to highlight that the variational formula (2.12) can be solved numerically to determine the variational phase in a relatively straightforward manner. We illustrate this by considering an explicit example, namely, a modified version of the Morris–Lecar model of a neuron [20]. The latter was originally introduced as a model of Ca^{2+} spikes in molluscs, but has subsequently been used to study neural excitability for Na^+ spikes [9], since it exhibits many of the same bifurcation scenarios as more complex models. Here we consider a stochastic version of the model that has been used to study subthreshold membrane potential oscillations (STOs) due to persistent sodium (Na^+) currents [28, 5].

The SDE for the membrane voltage v and recovery variable w (representing the fraction of open potassium (K^+) channels) evolves as

$$\begin{aligned}
dv_t &= \Big[a_\infty(v)f_{Na}(v) + wf_K(v) + f_L(v) + I_{\text{app}}\Big]dt + \sqrt{\epsilon}dW_t, \\
dw_t &= \Big[(1-w)\alpha_K(v) - w\beta_K\Big]dt,
\end{aligned} \tag{4.1}$$

where W_t is a Brownian motion. It is assumed that the Na^+ channels are in quasi-steady-state so that the fraction of open Na^+ ion channels is given by the voltage-dependent function

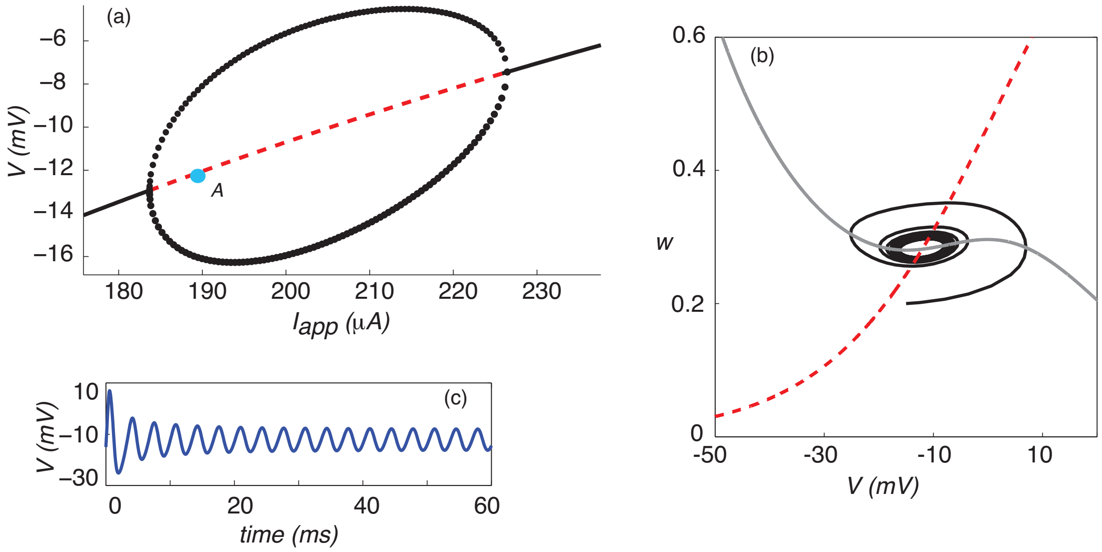

Figure 4.1. (a) *Bifurcation diagram of the deterministic model. As I_{app} is increased, the system undergoes a supercritical Hopf bifurcation (H) at $I^*_{\text{app}} = 183$, which leads to the generation of stable oscillations. The maximum and minimum values of oscillations are plotted as black (solid) curves. Oscillations disappear via another supercritical Hopf bifurcation.* (b) *Phase plane diagram of the deterministic model for $I_{\text{app}} = 190$ μA (point A above the Hopf bifurcation point). The red (dashed) curve is the w-nullcline and the solid (gray) curve represents the v-nullcline.* (c) *Corresponding voltage time courses.* Sodium parameters*: $g_{Na} = 4.4$ mS, $V_{Na} = 55$ mV, $\overline{\alpha}_{Na} = 100$ ms^{-1}, $v_{n,1} = -1.2$mV, $v_{n,2} = 18$ mV.* Leak parameters*: $g_L = 2$ mS, $V_L = -60$ mV.* Potassium parameters*: $g_K = 8$ mS, $V_K = -84$ mV, $\overline{\alpha}_K = 0.35$ ms^{-1}, $v_{k,1} = 2$ mV, $v_{k,2} = 30$ mV.*

$a_\infty(v)$, thus eliminating Na$^+$ as a variable. For $i = K, Na, L$, let $f_i = g_i(V_i - v)$, where g_i are ion conductances and V_i are reversal potentials. For concreteness, we take

$$\alpha_i(v) = \overline{\alpha}_i \exp\left(\frac{2(v - v_{i,1})}{v_{i,2}}\right), \quad i = K, Na, \tag{4.2}$$

with $\overline{\alpha}_i, v_{i,1}, v_{i,2}$ constant. Parameters are chosen such that stable oscillations arise for sufficient values of the applied current via a supercritical Hopf bifurcation (see Figure 4.1(a)). This corresponds well to observed behavior of STOs and is not meant to function as a traditional spiking neuron model. Limit cycles in a traditional spiking model often appear via a subcritical Hopf bifurcation. Figures 4.1(b),(c) show oscillations corresponding to point A in the bifurcation diagram.

Results from simulations of the stochastic Morris–Lecar model (4.1) for $\epsilon = 0.1$ are shown in Figure 4.2. In particular, we compare the linearized phase $\theta_t - t\omega_0$ with the exact variational phase $\beta_t - t\omega_0$ obtained from (2.15). It can be seen that initially the phases are very close, and then they slowly drift apart as noise accumulates. The diffusive nature of the drift in both phases can be clearly seen, with the typical deviation of the phase from $\omega_0 t$ increasing in time. The stable attractor of the deterministic limit cycle is quite large, which is why the system can tolerate quite substantial stochastic perturbations. The fact that, even with relatively large ϵ, the system stays close to the limit cycle for a very long time is a key motivation for the exponential probability bounds that we outline in section 5.

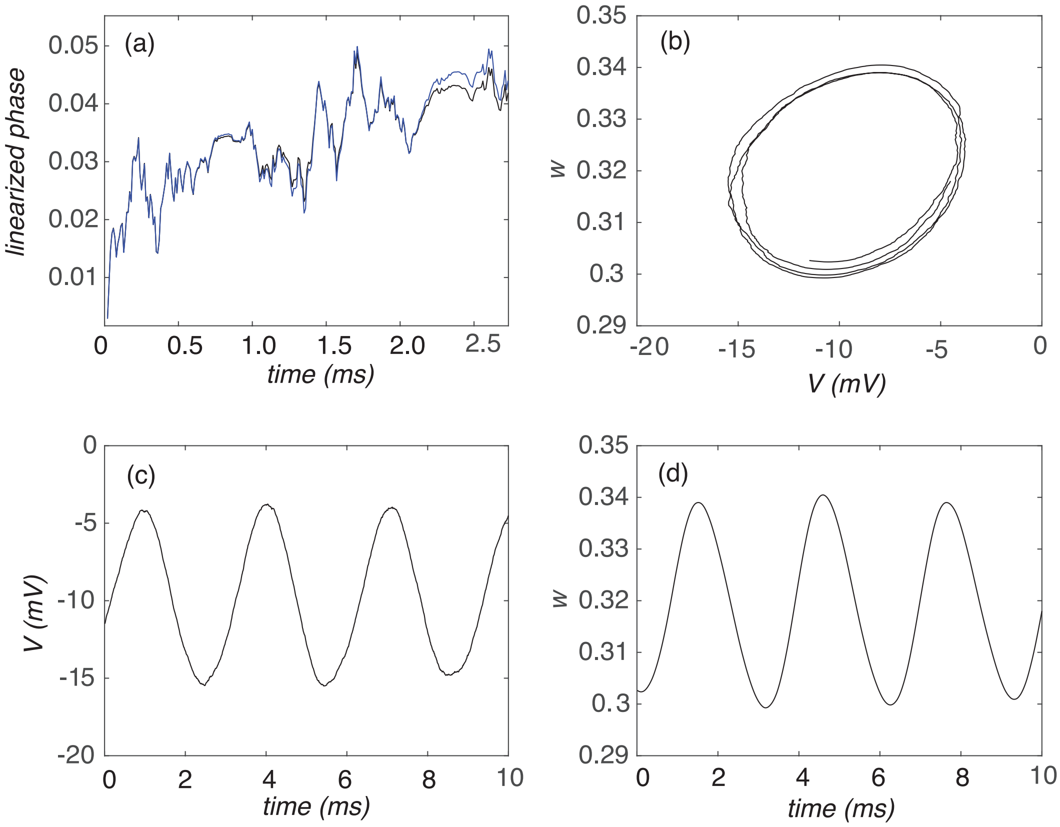

Figure 4.2. *We simulate the stochastic Morris–Lecar model with* $\epsilon = 0.1$. (a) *Plot of the linearized phase* $\theta_t - t\omega_0$ *in blue, and the exact variational phase (satisfying* (2.15)*)* $\beta_t - t\omega_0$ *in black.* (b) *Stochastic trajectory around limit cycle (dashed curve) in the* v, w*-plane.* (c,d) *Corresponding time variations in* v *and* w.

4.1. Some perspectives on the numerical implementation. One of the major advantages of the variational approach to phase reduction introduced in this paper is that it is both easier to implement, and also more numerically efficient than decompositions based on the isochronal phase. Indeed, evaluating the full nonlinear isochronal phase is known to be numerically expensive [2]. Furthermore, if one wishes to implement an SDE for the isochronal phase (which one could obtain through applying Ito's lemma to the isochronal phase map) then one would require the first and second derivatives of the isochronal phase. In most cases these would be numerically expensive to calculate [2] and, in almost all cases, there is not an analytical expression. The variational phase perspective is particularly flexible, since it is straightforward to switch between the "Hamiltonian" perspective of (2.12), and the "Lagrangian" dynamical perspective of (2.32). One can calculate the nonlinear variational phase in two different ways: either through taking a solution x_t, and directly minimizing (2.12), or through implementing the nonlinear SDE of β_t using (2.32). Before one performs either of these calculations, one must first calculate the Floquet matrix. This is relatively simple, since one just has to solve the ODE in (2.5) d times, with different starting conditions, and then combine the two solutions. One can then use this solution to determine the variational phase. It should be noted that even if one were implementing the simplest of all phase equations, i.e., the linearized isochronal SDE, then one would still need to calculate the Floquet decomposition to obtain $R(\theta)$. Thus there is nothing unreasonable in our requirement of the Floquet decomposition. If one wishes to determine the variational phase through minimizing (2.12), then a simple optimization routine suffices. We used the MATLAB nonlinear optimization package, and we achieved an accuracy of $O(10^{-5})$ using $O(10)$ steps.

5. Explicit bounds on the growth of the weighted amplitude $\|u_t - \Phi(\beta_t)\|_{\beta_t}$. In this section we obtain powerful bounds on how long it takes the weighted amplitude of the orthogonal fluctuations, $\|u_t - \Phi(\beta_t)\|_{\beta_t}$, to exceed some value a. These bounds are valid for $\|u_t - \Phi(\beta_t)\|_{\beta_t} = o(b)$, where b is the magnitude of the decay of transverse fluctuations, and are useful in a variety of situations. Since we are looking at the asymptotic regime of certain variables, it first needs to be noted which constants are taken to be $O(1)$: it is these constants which determine the constants C_1, C_2, and C below. These $O(1)$ constants are

$$\sup_{\theta\in[0,2\pi]} \|\Phi(\theta)\|, \sup_{\theta\in[0,2\pi]} \|P(\theta)\|, \sup_{\theta\in[0,2\pi]} \left\|P^{-1}(\theta)\right\|, \sup_{\theta\in[0,2\pi]} \left\|P'(\theta)\right\|, \|Q\|, \sup_{z\in\mathbb{R}^d} \|G(z)\|. \tag{5.1a}$$

Since $\Phi'(\theta)$ is the first column of $P(\theta)$, the above conditions also imply that

$$\sup_{\theta\in[0,2\pi]} \left\|\Phi'(\theta)\right\|, \quad \sup_{\theta\in[0,2\pi]} \left\|\Phi''(\theta)\right\| = O(1). \tag{5.1b}$$

Two regimes where our bounds are particularly useful are (i) the small noise limit ($\epsilon \to 0$) and (ii) a finite noise regime in which there is a large decay of fluctuations that are transverse to the limit cycle (i.e., large b) [26, 23]. These bounds are more powerful and flexible than classical large deviations bounds, because both the neighborhood $[0, a]$ and the time interval T can vary with ϵ and b. The relative simplicity of the proof of this theorem provides further justification for the phase decomposition outlined in the first half of this paper. It results in a uniform lower bound for the decay of the transformed drift $w_t = P\big(\beta_t\big)^{-1} v_t$, which means that after a rescaling of time using the Dambis–Dubins–Schwarz theorem [24], it becomes straightforward to demonstrate that the amplitude term behaves like a stable Ornstein–Uhlenbeck process. This theorem can also be used to bound the probability of the stopping time τ (defined in (2.18)) exceeding a certain value.

The following bounds are expressed in terms of the first hitting time of the scalar Ornstein–Uhlenbeck process, which we restate here. Let $p^{(-b)}_{x,a}(t)$ be the density for the first hitting time of the Ornstein–Uhlenbeck process with drift gradient $-b$ started at x. More precisely, if

$$dY_t = -\, bY_t dt + dW_t, \quad Y_0 = x, \tag{5.2}$$

for a one-dimensional Brownian motion W, then

$$\mathbb{P}\bigg(\inf\{s > 0 : Y_s = \kappa\} \in [t, t + dt]\bigg) := p^{(-b)}_{x,\kappa}(t)dt. \tag{5.3}$$

Let $I(\epsilon, b) \subset \mathbb{R}^+$ be the following closed interval

$$I(\epsilon, b) = \Big\{ a \in \mathbb{R}^+ : C_1\epsilon + C_2\big(1 + \sup_{z\in\mathbb{R}^d} \big\|F''(z)\big\|\big)a^2 \le \frac{ba}{2} \text{ and }$$
$$a \le \frac{1}{2} \sup_{\alpha\in[0,2\pi]} \left\| \Phi''(\alpha) - \frac{d}{d\theta}\big[P(\theta)P^\top(\theta)\big]\big|_{\theta=\alpha} P^{-\top}(\alpha)P^{-1}(\alpha)\Phi'(\alpha) \right\|_\alpha^{-1} \Big\}, \tag{5.4}$$

where C_1 and C_2 are positive constants (independent of ϵ and b) that are specified in Lemma A.1. These constants can be written as a function of the $O(1)$ parameters in (5.1a) and (5.1b). The second condition in the above definition is to ensure that the SDE for β_t is well-defined as long as $\|u_t - \Phi(\beta_t)\|_{\beta_t} \in I(\epsilon, b)$.

The following theorem obtains bounds on how long it takes $\|u_t - \Phi(\beta_t)\|_{\beta_t}$ to attain any a in $I(\epsilon, b)$. The theorem is most useful in the regime

$$a \in \left[O\left(\sqrt{\frac{\epsilon}{b}}\right), O\left(b / \sup_{z\in\mathbb{R}^d} \|F''(z)\|\right)\right],$$

where $\|F''(z)\|$ is the Euclidean norm of the tensor F''. It is less useful for values of a towards the lower end of $I(\epsilon, b)$, since $\|u_t - \Phi(\beta_t)\|_{\beta_t}$ will very quickly attain $O\big(\sqrt{\epsilon/b}\big)$, since in this regime the fluctuations of the noise dominate the $-b$ decay resulting from the stability of the deterministic dynamics.

Recall that τ (defined in (2.18)) is the stopping time such that the SDE for the phase in section 2 is well-defined for all $t \le \tau$.

Theorem 5.1. *For all $a \in I(\epsilon, b)$, if*

$$\sup_{s\in[0,T]} \|u_s - \Phi(\beta_s)\|_{\beta_s} \le a, \tag{5.5}$$

then $T \le \tau$. Furthermore, if $\|u_0 - \Phi(\beta_0)\|_{\beta_0} := x < \frac{a}{2}$, then

$$\mathbb{P}\left(\sup_{s\in[0,T]} \|u_s - \Phi(\beta_s)\|_{\beta_s} \ge a \right) \le \int_0^T p^{(-b)}_{\bar{x},\bar{a}}(u)du, \tag{5.6}$$

where $\bar{x} = x/\sqrt{\lambda\epsilon}$ and $\bar{a} = a/2\sqrt{\lambda\epsilon}$, and λ is a positive constant that is given in (A.22). Note that λ is determined by Π, G, and Q.

Remark 1. To facilitate the exposition, we have chosen $\bar{a} = a/2\sqrt{\lambda\epsilon}$. In fact, we could have chosen $\bar{a} = \rho a/\sqrt{\lambda\epsilon}$ for any $\rho \in (0, 1)$, and the bound would still hold in the limit $\epsilon/b \to 0$.

Remark 2. We can use classical results on the first hitting time of the Ornstein–Uhlenbeck process to derive the leading order asymptotics of the above [25, 1]. To leading order, as $b/\epsilon \to \infty$,

$$p^{(-b)}_{0,\bar{a}}(t) \simeq bg\left(\frac{a^2 b}{4\lambda\epsilon}\right) \exp\left\{ -btg\left(\frac{a^2 b}{4\lambda\epsilon}\right)\right\}, \tag{5.7}$$

where $g(z) = \frac{\sqrt{z}}{\sqrt{2\pi}} \exp\{-\frac{z}{2}\}$. We find that for $a \in I(\epsilon, b)$, if $T = o(g\{\frac{a^2 b}{4\lambda\epsilon}\}^{-1})$, then

$\mathbb{P}(\sup_{s\in[0,T]} \|u_s - \Phi(\beta_s)\|_{\beta_s} \geq a) \simeq Tg\{\frac{a^2 b}{4\lambda\epsilon}\} \ll 1$. There are much more refined estimates in the literature: note in particular the exact analytic expression in [1, Theorem 3.1].

6. Discussion and future work. In summary, the variational approach developed in this paper determines the phase of a stochastic oscillator by requiring it to minimize a weighted norm. We have demonstrated that, to leading order, the phase separates from the amplitude and agrees with the isochronal phase. Hence, the linearization of our phase dynamics is accurate over timescales of $O(\epsilon^{-1})$, which is the same order of accuracy as the isochronal phase equation. In addition, our exact phase equation (2.32) is accurate over much longer timescales of order $O\big(\exp(Cb\epsilon^{-1})\big)$, recalling that b is the rate of decay of transverse fluctuations. There exists a precise analytic expression for the phase SDE, as well as a stopping time τ up to which this SDE applies. Furthermore, one can immediately determine the phase from any particular realization of the fundamental SDE using (2.15) (as long as one takes the phase to be the global minimum). This is an advantage of our method compared to the isochronal method, since in most cases there does not exist an analytic solution for the isochronal method, and it is difficult to implement in a computationally efficient way [2].

We noted in the introduction that Giacomin, Poquet, and Shapira [12, Lemma 4.1, 4.2] also bound the probability of the system escaping a neighborhood of the limit cycle. They use a different method than us, approximating the rate of change of the phase to be constant over the course of one cycle. In the regime $\epsilon \to 0$, their results are comparable to ours. However their bound is much less accurate than ours in the regime $b \to \infty$, since their bound predicts that the system will stay in a neighborhood of the limit cycle over times of order $\exp(-C\epsilon^{-1}\exp(-4b\omega_0^{-1}))$, where $2\pi\omega_0^{-1}$ is the period of the limit cycle. Also, ours is in general more accurate for a limit cycle with large period (i.e., small ω_0), as long as the variables in (5.1a) and (5.1b) remain $O(1)$. This comes from our particular choice of phase, since we do not need to assume that the rate of change of the phase is approximately constant over the course of one limit cycle. Herein lies the advantage of continually and dynamicalIy adjusting the phase. Indeed the accuracy of our method for this regime can be seen from our simulated results. Since ω_0 is quite small relative to ϵ, the bound $\exp(-C\epsilon^{-1}\exp(-4b\omega_0^{-1}))$ of [12] is not particularly powerful but, on the other hand, our results demonstrate that the system stays in a neighborhood of the limit cycle for a much longer time than their result might predict. To be fair to [12], the aim of their paper is a little different from ours: they are primarily looking at the asymptote of the winding number in the limit as $\epsilon \to 0$.

The phase SDE (2.32) is thus very well suited to studying the long-time dynamics of the phase on timescales of $O\big(\exp(Cb\epsilon^{-1})\big)$. In section 5 we obtained powerful bounds on the probability of the oscillator leaving any particular neighborhood of the oscillator over any particular timescale. These bounds are very flexible, because they shed light on the mutual scaling of the amplitude of the noise, rate of decay of transverse fluctuations, the size of the neighborhood of the limit cycle, the period of the limit cycle, and the time the oscillator spends in this neighborhood.

In forthcoming work, we will use the phase SDE of this paper to study the synchronization of uncoupled oscillators subject to common noise. In particular, we will obtain precise bounds on the probability of two synchronized oscillators desynchronizing, and conditions under which two oscillators never desynchronize. Another interesting application of the phase SDE of this paper would be the effect of finite noise on oscillators with a strong decay of transverse fluctuations [23].

Appendix A. Proof of Theorem 5.1.

Proof. We start with the first part of the theorem. From (2.17),

$$\begin{aligned}\mathcal{H}(z,\theta) &= 1 - \Big\langle z-\Phi(\theta), \Phi''(\theta)\Big\rangle_\theta - \Big\langle z-\Phi(\theta), P(\theta)P^\top(\theta)\frac{d}{d\theta}\big[P(\theta)P^\top(\theta)\big]^{-1}\Phi'(\theta)\Big\rangle_\theta \\ &= 1 - \Big\langle z-\Phi(\theta), \Phi''(\theta) - \frac{d}{d\theta}\big[P(\theta)P^\top(\theta)\big]\big[P(\theta)P^\top(\theta)\big]^{-1}\Phi'(\theta)\Big\rangle_\theta,\end{aligned}$$

and through an application of the Cauchy–Schwarz inequality to the above, it may be observed that

$$\mathcal{H}(u_t,\beta_t) \geq 1 - \|u_t-\Phi(\beta_t)\|_{\beta_t}\left\|\Phi''(\beta_t) - \frac{d}{d\theta}\big[P(\theta)P^\top(\theta)\big]\big|_{\theta=\beta_t}P^{-\top}(\beta_t)P^{-1}(\beta_t)\Phi'(\beta_t)\right\|_{\beta_t}.$$

It then follows from the definition of $I(\epsilon,b)$ that if $\sup_{s\in[0,t]}\|u_s-\Phi(\beta_s)\|_{\beta_s} \leq a$ for $a \in I(\epsilon,b)$, then $\mathcal{H}(u_s,\beta_s) \geq \frac{1}{2}$ for all $s \in [0,t]$ and, therefore,

$$\sup_{s\in[0,t]} \mathcal{H}(u_s,\beta_s)^{-1} \leq 2. \tag{A.1}$$

This means that

$$\tau \geq \inf\big\{s \geq 0 : \|u_s-\Phi(\beta_s)\|_{\beta_s} = a\big\}. \tag{A.2}$$

In other words, the SDE for the phase β_t that we derived in section 2 is well-defined as long as $\|u_t-\Phi(\beta_t)\|_{\beta_t} \leq a$.

We now prove the second part of the theorem. Recall that the amplitude term satisfies (2.35). In the following it is convenient to perform the rescaling $\sqrt{\epsilon}v_t \to v_t$:

$$dv_t = \big[J\big(\beta_t\big)v_t + \gamma_0(u_t,\beta_t)\big]\,dt + \sqrt{\epsilon}\widetilde{G}(u_t,\beta_t)dW_t, \tag{A.3}$$

where

$$\begin{aligned}\gamma_0(u_t,\beta_t) = F(u_t) - J(\beta_t)v_t - \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)\Big(\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t} + \epsilon\kappa(u_t,\beta_t)\Big) \\ - \frac{\epsilon}{2}\Phi''(\beta_t)\mathcal{H}(u_t,\beta_t)^{-2}\big\langle K(u_t,\beta_t)\Phi'(\beta_t), QK(u_t,\beta_t)\Phi'(\beta_t)\big\rangle\end{aligned} \tag{A.4}$$

and $\widetilde{G}(u_t,\beta_t) \in \mathbb{R}^{d\times d}$ is given by

$$\widetilde{G}(u_t,\beta_t) = G(u_t) - \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)\Phi'(\beta_t)^\top P^{-\top}\big(\beta_t\big)P^{-1}\big(\beta_t\big)G(u_t).$$

We now perform the change of variable $w_t = P^{-1}(\beta_t)v_t$, since $\|v_t\|_{\beta_t} = \|w_t\|$. Using Ito's lemma, we find that

$$\text{(A.5)}\quad dw_t = \frac{1}{2}\frac{d^2}{d\beta_t^2}P^{-1}(\beta_t)v_t d\beta_t d\beta_t - P^{-1}\big(\beta_t\big)P'\big(\beta_t\big)w_t d\beta_t + P^{-1}\big(\beta_t\big)dv_t - P^{-1}\big(\beta_t\big)P'\big(\beta_t\big)P^{-1}\big(\beta_t\big)dv_t d\beta_t.$$

As will be seen further below, the reason for this change of variable is that the drift of w_t decays uniformly (to leading order), so that the leading order behavior of the SDE is like a stable Ornstein–Uhlenbeck process. We now demonstrate this. Recall from (2.10) that the derivative of $P(t)$ satisfies

$$\text{(A.6)}\qquad \omega_0 P'(\theta) = J(\theta)P(\theta) - P(\theta)\mathcal{S}.$$

This means that

$$dw_t = \frac{1}{2}\frac{d^2}{d\beta_t^2}P^{-1}(\beta_t)v_t d\beta_t d\beta_t + \omega_0^{-1}\big(- P^{-1}\big(\beta_t\big)J(\beta_t)v_t + \mathcal{S}w_t\big)d\beta_t + P^{-1}\big(\beta_t\big)dv_t - P^{-1}\big(\beta_t\big)P'\big(\beta_t\big)P^{-1}\big(\beta_t\big)dv_t d\beta_t$$

and, therefore,

$$\text{(A.7)}\quad dw_t = \big[\mathcal{S}w_t + \gamma(u_t,\beta_t)\big]dt + \sqrt{\epsilon}P^{-1}(\beta_t)\widetilde{G}(u_t,\beta_t)dW_t + \sqrt{\epsilon}\omega_0^{-1}\mathcal{H}(u_t,\beta_t)^{-1}\big\{ - P^{-1}\big(\beta_t\big)J(\beta_t)v_t + \mathcal{S}w_t\big\}\Phi'(\beta_t)^\top P^{-\top}(\beta_t)P^{-1}(\beta_t)G(u_t)dW_t,$$

where

$$\text{(A.8)}\quad \gamma(u_t,\beta_t) = \frac{1}{2}\frac{d^2}{d\beta_t^2}P^{-1}(\beta_t)v_t d\beta_t d\beta_t - \omega_0^{-1}P^{-1}(\beta_t)\big(J(\beta_t)v_t + \gamma_0(u_t,\beta_t)\big) - \mathcal{S}w_t + \omega_0^{-1}\big(- P^{-1}\big(\beta_t\big)J(\beta_t)v_t + \mathcal{S}w_t\big)\mathcal{H}(u_t,\beta_t)^{-1}\left(\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t} + \epsilon\kappa(u_t,\beta_t)\right) - \epsilon\mathcal{H}(u_t,\beta_t)^{-1}P^{-1}\big(\beta_t\big)P'\big(\beta_t\big)P^{-1}\big(\beta_t\big)\widetilde{G}\big(u_t,\beta_t\big)QG^\top(u_t)P^{-\top}(\beta_t)P^{-1}(\beta_t)\Phi'(\beta_t)$$

and we have used the fact that

$$\begin{aligned} d\beta_t &= \sqrt{\epsilon}\mathcal{H}(u_t,\beta_t)^{-1}\big\langle P^{-1}(\beta_t)\Phi'(\beta_t), P^{-1}(\beta_t)G(u_t)dW_t\big\rangle + F.V.T \\ &= \sqrt{\epsilon}\mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)^\top P^{-\top}(\beta_t)P^{-1}(\beta_t)G(u_t)dW_t + F.V.T, \end{aligned}$$

where $F.V.T$ stands for "finite variation terms" (i.e., the drift terms). We write this as

$$\text{(A.9)}\qquad dw_t = \Big[\mathcal{S}w_t + \gamma(u_t,\beta_t)\Big]dt + \sqrt{\epsilon}\bar{G}(u_t,\beta_t)dW_t,$$

where $\bar{G}(u_t,\beta_t)$ can be inferred from (A.7).

Since the map $w \to \|w\|^2$ is twice differentiable, we can apply Ito's lemma to (A.9). We find that

$$d\|w_t\|^2 = \Big[2\big\langle w_t, \mathcal{S}w_t + \gamma(u_t,\beta_t)\big\rangle + \epsilon \mathrm{tr}\big\{\bar{\mathrm{G}}(\mathrm{u_t},\beta_\mathrm{t})\mathrm{Q}\bar{\mathrm{G}}^\top(\mathrm{u_t},\beta_\mathrm{t})\big\}\Big]\mathrm{dt} + 2\sqrt{\epsilon}\Big\langle w_t, \bar{G}(u_t,\beta_t)dW_t\Big\rangle. \tag{A.10}$$

It follows from the stability assumption at the start of this paper that $\langle w_t, \mathcal{S}w_t\rangle \le -b\|w_t\|^2$, which means that

$$d\|w_t\|^2 \le \big[-2b\|w_t\|^2 + 2\gamma_2(u_t,\beta_t)\big]dt + 2\sqrt{\epsilon}\langle w_t, \bar{G}(u_t,\beta_t)dW_t\rangle, \tag{A.11}$$

where

$$\gamma_2(u_t,\beta_t) = \langle w_t, \gamma(u_t,\beta_t)\rangle + \frac{\epsilon}{2}\mathrm{tr}\big\{\bar{\mathrm{G}}(\mathrm{u_t},\beta_\mathrm{t})\mathrm{Q}\bar{\mathrm{G}}^\top(\mathrm{u_t},\beta_\mathrm{t})\big\}.$$

Define the stopping time

$$\hat{\tau}_a = \inf\Big\{s \le \tau : \|w_s\|^{-1}\gamma_2\big(u_s,\beta_s\big) = ba/2\Big\}, \tag{A.12}$$

recalling that τ (defined in (2.18)) is the stopping time for which the SDE for β_t is well-defined.

We determine an SDE for $\|w_t\|$ by applying Ito's lemma to the square root function, finding that for all times $t \le \hat{\tau}_a$

$$\begin{aligned} d\|w_t\| &\le \sqrt{\epsilon}\|w_t\|^{-1}\langle w_t, \bar{G}(u_t,\beta_t)dW_t\rangle \\ &\quad + \Big(-b\|w_t\| + \|w_t\|^{-1}\gamma_2(u_t,\beta_t) - \frac{\epsilon}{2\|w_t\|^3}\big\langle Q\bar{G}^\top(u_t)w_t, \bar{G}^\top(u_t,\beta_t)w_t\big\rangle\Big)dt \\ &\le \sqrt{\epsilon}\|w_t\|^{-1}\langle w_t, \bar{G}(u_t,\beta_t)dW_t\rangle + \Big(-b\|w_t\| + \|w_t\|^{-1}\gamma_2(u_t,\beta_t)\Big)dt, \end{aligned} \tag{A.13}$$

since $\frac{\epsilon}{4\|w_t\|^3}\big\langle Q\bar{G}^\top(u_t,\beta_t)w_t, \bar{G}^{\top,\beta_t}(u_t)w_t\big\rangle \ge 0$, because the covariance matrix Q is positive semi-definite. Note that the coefficients of the above SDE are continuous and bounded in a sufficiently small neighborhood of $\|w_t\| = 0$. This is true for $\|w_t\|^{-1}\gamma_2$ thanks to the inequality in Lemma A.1, and it is true for the diffusion term thanks to the Cauchy–Schwarz inequality (this will be clear in the following).

Now define $y_t = \exp\big(bt\big)\|w_t\|$. Through Ito's lemma, we find that

$$dy_t = by_t dt + \exp\big(bt\big)d\|w_t\|$$

and, therefore, for all times $t \le \hat{\tau}_a$,

$$dy_t \le \exp\big(bt\big)\Big\{b\|w_t\| - b\|w_t\| + \|w_t\|^{-1}\gamma_2(u_t,\beta_t)\Big\}dt + \sqrt{\epsilon}\|w_t\|^{-1}\exp\big(bt\big)\big\langle w_t, \bar{G}(u_t,\beta_t)dW_t\big\rangle.$$

We integrate the above expression, before dividing both sides by $\exp(bt)$, and find that

$$\|w_{t\wedge\hat{\tau}_a}\| \leq \exp\left\{-b(t\wedge\hat{\tau}_a)\right\}\|w_0\| + \sqrt{\epsilon}\int_0^{t\wedge\hat{\tau}_a}\exp\left(b(s-t\wedge\hat{\tau}_a)\right)\|w_s\|^{-1}\left\langle w_s, \bar{G}(u_s,\beta_s)dW_s\right\rangle + \int_0^{t\wedge\hat{\tau}_a}\exp\left(b(s-t\wedge\hat{\tau}_a)\right)\|w_s\|^{-1}\gamma_2(u_s,\beta_s)ds.$$

This means that

$$\text{(A.14)}\quad \|w_{t\wedge\hat{\tau}_a}\| \leq \exp\left\{-b(t\wedge\hat{\tau}_a)\right\}\|w_0\| + \frac{1}{b}\sup_{s\in[0,t\wedge\hat{\tau}_a]}\|w_s\|^{-1}\left|\gamma_2(u_s,\beta_s)\right| + \sqrt{\epsilon}\int_0^{t\wedge\hat{\tau}_a}\|w_s\|^{-1}\exp\left(b(s-t\wedge\hat{\tau}_a)\right)\left\langle w_s,\bar{G}(u_s,\beta_s)dW_s\right\rangle$$
$$\leq \exp\left\{-b(t\wedge\hat{\tau}_a)\right\}\|w_0\| + \frac{a}{2} + \sqrt{\epsilon}\int_0^{t\wedge\hat{\tau}_a}\|w_s\|^{-1}\exp\left\{b(s-t\wedge\hat{\tau}_a)\right\}\left\langle w_s,\bar{G}(u_s,\beta_s)dW_s\right\rangle,$$

using the definition of $\hat{\tau}_a$.

Define the stopping time

$$\text{(A.15)}\qquad \tau_{a,x} = \inf\left\{\hat{\tau}_a, \check{\tau}_{a,x}\right\},$$

where

$$\text{(A.16)}\qquad \check{\tau}_{a,x} = \inf\left\{s\geq 0 : x\exp\left(-bs\right) + \sqrt{\epsilon}\int_0^s\|w_t\|^{-1}\exp\left(b(t-s)\right)\left\langle w_t,\bar{G}(u_t,\beta_t)dW_t\right\rangle = a/2\right\}.$$

It follows from (A.14) that for all $s\in[0,\tau_{a,x}]$,

$$\text{(A.17)}\qquad \|w_s\|\leq a.$$

This means that

$$\text{(A.18)}\quad \mathbb{P}\left(\tau_{a,x}\leq T\right) \leq \mathbb{P}\Bigg(\text{There exists } s\in[0,T] \text{ such that either } \zeta_s - x \geq \exp(bs)\frac{a}{2} \text{ or } \left|\frac{1}{\|w_s\|}\gamma_2(u_s,\beta_s)\right| = ba/2, \text{ and } \sup_{r\in[0,s]}\|w_r\|\leq a\Bigg)$$
$$\leq \mathbb{P}\Big(\text{There exists } s\in[0,T] \text{ such that } \zeta_s - x\geq \exp(bs)a/2\Big) + \mathbb{P}\Big(\text{There exists } s\in[0,T] \text{ such that } \left|\gamma_2(u_s,\beta_s)\right| = ba/2 \text{ and } \sup_{r\in[0,s]}\|w_r\|\leq a\Big),$$

where $\zeta_s = \sqrt{\epsilon}\int_0^s\|w_t\|^{-1}\exp\left(bt\right)\left\langle w_t,\bar{G}(u_t,\beta_t)dW_t\right\rangle$.

Now it follows from (A.2) that

$$\mathbb{P}\left(\tau\leq T \text{ and } \sup_{s\in[0,T]}\|w_s\|\leq a\right) = 0.$$

Furthermore, it follows from Lemma A.1 that

$$\begin{aligned}&\mathbb{P}\Big(\text{There exists } s \in [0,\tau] \text{ such that } \big|\, \|w_s\|^{-1} \gamma_2(u_s,\beta_s)\big)\big| = ba/2 \text{ and } \sup_{t\in[0,s]} \|w_t\| \leq a\Big) \\ &\quad \leq \mathbb{P}\Big(\text{There exists } s \in [0,\tau] \text{ such that } C_1\epsilon + C_2 \|w_s\|^2 = ba/2 \text{ and } \sup_{t\in[0,s]} \|w_t\| \leq a\Big) \\ &\quad = 0,\end{aligned}$$

thanks to the fact that $a \in I(\epsilon, b)$, which we recall is defined in (5.4).

It therefore remains for us to prove that

$$\mathbb{P}\Big(\text{There exists } s \in [0,T] \text{ such that } \zeta_s - x \geq \exp(bs)\frac{a}{2}\Big) \leq \int_0^T p^{(-b)}_{\bar{x},\bar{a}}(y)dy, \tag{A.19}$$

recalling that $\bar{a} = a/2\sqrt{\lambda\epsilon}$ and $\bar{x} = x/\sqrt{\lambda\epsilon}$.

By the Dambis–Dubins–Schwarz theorem [24, Theorem 1.6, Page 181], $X_t := \zeta_{\iota_t}$ is Brownian, where

$$\iota_s = \inf\big\{r \geq 0 : \eta_r \geq s\big\} \text{ and} \tag{A.20}$$

$$\eta_r := \epsilon \int_0^r \|w_s\|^{-2} \exp\left(2bs\right)\big\langle \bar{G}^\top(u_s,\beta_t)w_s, Q\bar{G}^\top(u_s,\beta_s)w_s\big\rangle ds. \tag{A.21}$$

Let $\lambda_{G,P}$ be an upper bound for $\big\|\bar{G}(u_t,\beta_t)\big\|$ (the spectral norm) that is uniform over all $\beta_t \in \mathbb{R}$ and $u_t \in \mathbb{R}^d$, recalling the implicit definition of $\bar{G}$ in (A.9). Such an upper bound exists, because by assumption $\|G(u_t)\|$ possesses a uniform upper bound. Similarly $P^{-1}(\beta_t)$ and $J(\beta_t)$ possess uniform upper bounds because they are continuous and 2π-periodic. Since $\bar{G}(u_t,\beta_t)$ is equal to sums and multiplications of matrices with uniform upper bounds, it must also possess a uniform upper bound. It follows that

$$\begin{aligned}\big\langle \bar{G}^\top(u_s,\beta_s)w_s, Q\bar{G}^\top(u_s,\beta_s)w_s\big\rangle &= \big\langle w_s, G(u_s,\beta_s)Q\bar{G}^\top(u_s,\beta_s)w_s\big\rangle \\ &\leq \lambda \|w_s\|^2,\end{aligned} \tag{A.22}$$

where $\lambda = \lambda^2_{G,P}\lambda_Q$. We find that $\eta_r \leq \bar{\eta}_r := \frac{\epsilon}{2b}\{\exp(br) - 1\}\lambda$. Writing $\bar{\iota}_s = \inf\{r \geq 0 : \bar{\eta}_r \geq s\}$, we have that $\bar{\iota}_s \leq \iota_s$ and

$$\begin{aligned}&\mathbb{P}\Big(\text{There exists } r \in [0,T]\,,\ \zeta_r - x \geq \exp(br)\frac{a}{2}\Big) \\ &= \mathbb{P}\Big(\text{There exists } r \in [0,T]\,,\ X_{\eta_r} - x \geq \exp(br)\frac{a}{2}\Big) \\ &= \mathbb{P}\Big(\text{There exists } y \in [0,\eta_T]\,,\ X_y - x \geq \exp(b\iota_y)\frac{a}{2}\Big) \\ &\leq \mathbb{P}\Big(\text{There exists } y \in [0,\bar{\eta}_T]\,,\ X_y - x \geq \exp\left(b\iota_y\right)\frac{a}{2}\Big) \\ &\leq \mathbb{P}\Big(\text{There exists } y \in [0,\bar{\eta}_T]\,,\ X_y - x \geq \exp\left(b\bar{\iota}_y\right)\frac{a}{2}\Big).\end{aligned} \tag{A.23}$$

Now suppose that Z satisfies the SDE

$$dZ_t = -bZ_t dt + \sqrt{\lambda\epsilon}\, dW_t, \tag{A.24}$$
$$Z_0 = x, \tag{A.25}$$

for a 1-dimensional Brownian motion W. The solution of this SDE is

$$Z_t = \exp\big(-bt\big)x + \sqrt{\lambda\epsilon}\int_0^t \exp\big(b(s-t)\big)dW_s. \tag{A.26}$$

Now let $\alpha_t = \sqrt{\lambda\epsilon}\int_0^t \exp\big(bs\big)dW_s$, and observe that the quadratic variation of α_t is $\bar{\eta}_t$. This means that

$$\mathbb{P}\Big(\sup_{t\in[0,T]} Z_t \geq \frac{a}{2}\Big) = \mathbb{P}\Big(\text{There exists } t\in[0,T] \text{ such that } \alpha_t + x \geq \frac{a}{2}\exp\big(bt\big)\Big)$$
$$= \mathbb{P}\Big(\text{There exists } t\in\big[0,\bar{\eta}_T\big] \text{ such that } \upsilon_t + x \geq \frac{a}{2}\exp\big(b\bar{\iota}_t\big)\Big) \tag{A.27}$$

by the Dambis–Dubins–Schwarz theorem, since $\upsilon_t := \alpha_{\bar{\iota}_t}$ is Brownian. It can be observed that the expressions in (A.27) and (A.23) are equal.

This means that

$$\mathbb{P}\Big(\text{There exists } t\in[0,T] \text{ such that } Z_t \geq \frac{a}{2}\Big)$$
$$= \mathbb{P}\Big(\text{There exists } y\in[0,\bar{\eta}^a_T] \text{ such that } X_y - \|u_0 - \Phi(\beta_0)\|_{\beta_0} \geq \exp(b\bar{\iota}_y)\frac{a}{2}\Big).$$

Now it can be seen that $\bar{Z}_t := \frac{1}{\sqrt{\lambda\epsilon}}Z_t$ is an Ornstein–Uhlenbeck process and, therefore,

$$\mathbb{P}\Big(\text{There exists } t\in[0,T] \text{ such that } Z_t \geq \frac{a}{2}\Big)$$
$$= \mathbb{P}\Big(\text{There exists } t\in[0,T] \text{ such that } \bar{Z}_t \geq \frac{a}{2\sqrt{\epsilon\lambda}}\Big) = \int_0^T p^{(-b)}_{\bar{x},\bar{a}}(s)ds,$$

and we have proved the required bound in (A.19). ∎

Lemma A.1. *There exist positive constants C_1 and C_2 such that, as long as $\|w_t\| \leq a \in I(\epsilon, b)$,*

$$\big|\gamma_2(u_t,\beta_t)\big| \leq C_1\|w_t\|\,\epsilon + C_2\Big(1 + \sup_{z\in\mathbb{R}^d}\big\|F''(z)\big\|\Big)\|w_t\|^3.$$

Proof. We can decompose $\gamma_2 = \gamma_2^1 + \epsilon\gamma_2^2$, where γ_2^1 comprises higher-order corrections to the linearized behavior, and γ_2^2 arises from quadratic and cross variations. In the following equations, since $v_t = P(\beta_t)^{-1}w_t$ and $P(\beta_t)^{-1}$ is continuous on $\mathbb{S}^1$, it must be the case that for some constant C_P, $\sup_{\theta\in[0,2\pi]} \left\|P^{-1}(\theta)\right\| \leq C_P$ and, therefore, $\|v_t\| \leq C_P \|w_t\|$. Using the definitions in (A.4), the higher-order corrections to the linearized behavior are

$$\begin{aligned}\gamma_2^1 = \Big\langle w_t, P^{-1}(\beta_t)\Big\{ F(u_t) - \mathcal{H}(u_t,\beta_t)^{-1}\Phi'(\beta_t)\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t} \\ - J\big(\beta_t\big)v_t\mathcal{H}(u_t,\beta_t)^{-1}\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t}\Big\}\Big\rangle \\ + \omega_0^{-1}\big\langle w_t,\mathcal{S}w_t\big\rangle\Big(\omega_0 - \mathcal{H}(u_t,\beta_t)^{-1}\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t}\Big),\end{aligned}$$

and the quadratic/cross-variation terms are

$$\begin{aligned}\gamma_2^2 = -\kappa(u_t,\beta_t)\mathcal{H}(u_t,\beta_t)^{-1}\Big\langle w_t, P^{-1}(\beta_t)\Phi'(\beta_t)\Big\rangle \\ - \frac{\mathcal{H}(u_t,\beta_t)^{-2}}{2}\Big\langle K(u_t,\beta_t)\Phi'(\beta_t), QK(u_t,\beta_t)\Phi'(\beta_t)\Big\rangle\Big\langle w_t, P^{-1}(\beta_t)\Phi''(\beta_t)\Big\rangle \\ + \mathcal{H}(u_t,\beta_t)^{-1}\Big\langle w_t, P^{-1}\big(\beta_t\big)P'\big(\beta_t\big)P^{-1}\big(\beta_t\big)\widetilde{G}\big(u_t,\beta_t\big)QG^\top(u_t,\beta_t)P^{-\top}(\beta_t)P^{-1}(\beta_t)\Phi'(\beta_t)\Big\rangle \\ + \frac{1}{2}\mathrm{tr}\big\{\bar{\mathrm{G}}(\mathrm{u_t},\beta_\mathrm{t})\mathrm{Q}\bar{\mathrm{G}}^\top(\mathrm{u_t},\beta_\mathrm{t})\big\} \\ + \frac{1}{2}\frac{d^2}{d\beta_t^2}P^{-1}(\beta_t)v_t d\beta_t d\beta_t + \omega_0^{-1}\mathcal{H}(u_t,\beta_t)^{-1}\big(-P^{-1}(\beta_t)J(\beta_t)v_t + Sw_t\big)\kappa(u_t,\beta_t).\end{aligned} \tag{A.28}$$

We start by bounding the quadratic and cross-variation terms, i.e., γ_2^2. Now since, by assumption, $\|w_t\| \leq a \in I(\epsilon,b)$, it follows from (A.1) that

$$\mathcal{H}(u_s,\beta_s)^{-1} \leq 2. \tag{A.29}$$

Our assumption in (5.1a)–(5.1b) means that there are uniform bounds for $\|G(u_t)\|$, $\|Q\|$, $\|P(\beta_t)\|$, $\left\|P(\beta_t)^{-1}\right\|$, and $\|P'(\beta_t)\|$. It follows from this that there are uniform bounds for $\|\Phi'(\beta_t)\|$, $\|\Phi''(\beta_t)\|$.

We now turn to bounding γ_2^1. First, it follows from the uniform boundedness of P^{-1} that for some constant C_P,

$$\|w_t\| = \left\|P^{-1}(\beta_t)v_t\right\| \leq C_P\|v_t\|.$$

Now we saw in the equations following (3.6) that

$$\mathcal{H}(u_t,\beta_t)^{-1}\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t} = \omega_0 + O\big(\|v_t\|^2\big).$$

This means that

$$\big\langle w_t,\mathcal{S}w_t\big\rangle\Big(\omega_0 - \mathcal{H}(u_t,\beta_t)^{-1}\big\langle F(u_t),\Phi'(\beta_t)\big\rangle_{\beta_t}\Big) \simeq O\big(\|w_t\|^4\big).$$

It remains for us to show that $F(u_t) - J\big(\beta_t\big)v_t - \Phi'\big(\beta_t\big) = O(\|v_t\|^2)$. But this follows from the multivariate Taylor remainder theorem, since for some $\upsilon \in [0,1]$,

$$F\big(\Phi(\beta_t) + v_t\big) = F\big(\Phi(\beta_t)\big) + J\big(\beta_t\big)v_t + \frac{1}{2}F''\big(\upsilon\Phi(\beta_t) + (1-\upsilon)v_t\big)\cdot v_t\cdot v_t. \tag{A.30}$$

By assumption, the second derivative of F is uniformly bounded, and we have therefore obtained the required bound. ■

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