Homework 12: Picard, Mittag-Leffler, Weierstrass, CA meets CA

Picard theorems.

1. Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire.

   (i) If $f^2 + g^2 = 1$ show that there is an entire function h such that $f(z) = \cos(h(z))$ and $g(z) = \sin(h(z))$ for every $z \in \mathbb{C}$. Hint: If an entire function doesn't take value 0 then it is the exponential of another entire function. Also, $(f + ig)(f - ig) = 1$. This actually works for simply-connected domains, not just $\mathbb{C}$.

   (ii) If $f^n + g^n = 1$ for some $n \geq 3$ show that f and g are constant. Hint: Factor into n terms and use Picard.

2. Let $f_n : \mathbb{D} \to \mathbb{C} \smallsetminus \{0, 1\}$ be holomorphic. Show that after a subsequence either:

   - f_n converge uniformly on compact sets to a function $f : \mathbb{D} \to \mathbb{C} \smallsetminus \{0, 1\}$, or
   - f_n converge uniformly on compact sets to a constant $0, 1$ or ∞.

   Hint: As usual, view $\mathbb{C} \smallsetminus \{0, 1\} = \hat{\mathbb{C}} \smallsetminus \{0, 1, \infty\}$ with its hyperbolic metric (the double of an ideal hyperbolic triangle). Then pass to a subsequence so that either all $f_n(0)$ are contained in a fixed compact set, or so that $f_n(0)$ converge to one of three cusps.

3. Use the previous problem to prove the following theorem of Schottky. For every $M > 0$ there is $C > 0$ so that if $f : \mathbb{D} \to \mathbb{C} \smallsetminus \{0, 1\}$ is holomorphic and $|f(0)| \leq M$ then $|f(z)| \leq C$ for every z with $|z| \leq \frac{1}{2}$. Of course, $\frac{1}{2}$ could be replaced by any number < 1 but C depends on this number.

4. Use Schottky's theorem to give another proof of Great Picard using the following outline. Let $f : \mathbb{D} \smallsetminus \{0\} \to \mathbb{C} \smallsetminus \{0, 1\}$ have a singularity at 0.

   (i) Show that there is a sequence $z_n \to 0$ such that $|f(z_n)|$ is uniformly bounded by some M or else 0 is a pole. Assume the former.

(ii) When z_n is sufficiently close to 0 show that $|f(z)| \leq C$ on the circle $|z| = |z_n|$ where C is the constant from Schottky's theorem. Hint: Consider $w \mapsto f(z_n e^{2\pi i w})$ for $|w| \leq \frac{1}{2}$.

(iii) Use the maximum principle to show that f is bounded by C on every annulus between two such circles.

(iv) Conclude that 0 is a removable singularity.

Mittag-Leffler and Weierstrass.

5. Let $\Omega \subseteq \mathbb{C}$ be a domain and a_n a sequence of distinct points in Ω that doesn't accumulate anywhere in Ω. If b_n is any sequence of complex numbers, show that there is a holomorphic function $f : \Omega \to \mathbb{C}$ so that $f(a_n) = b_n$ for all n. Moreover, show that for each a_n we can specify the initial portion of the power series expansion around a_n. Hint: First use Weierstrass, then Mittag-Leffler. It's important to do it in that order. The desired function will be the product.

Complex Analysis meets Commutative Algebra: algebraic properties of the ring $H(\Omega)$.

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $H(\Omega)$ denote the ring of holomorphic functions $\Omega \to \mathbb{C}$. In this section you can use Mittag-Leffler, Weierstrass and the previous problem. The purpose is to establish some properties of this ring that you encounter in commutative algebra.

6. Let $f_1, f_2 \in H(\Omega)$ be two functions without common zeros. Show that there are functions $g_1, g_2 \in H(\Omega)$ such that $f_1 g_1 + f_2 g_2 = 1$. Hint: Choose g_1 so that $1 - f_1 g_1$ vanishes at every point where f_2 does with at least as large multiplicity.

7. More generally, show that for any $f_1, f_2 \in H(\Omega)$ there are $g_1, g_2 \in H(\Omega)$ so that $h = f_1 g_1 + f_2 g_2$ vanishes only at points where both f_1, f_2 vanish and the multiplicity of the zero is the smaller of the two multiplicities for f_1, f_2.

8. Show that every finitely generated ideal in $H(\Omega)$ is principal. Hint: For two generators this should follow from the previous problem. In general induct.

9. Show that every finitely generated maximal ideal I is of the form

$$\{f \in H(\Omega) \mid f(z_0) = 0\}$$

for some $z_0 \in \Omega$. This is a kind of *Nullstellensatz* in this setting.

10. Construct an (infinitely generated) ideal in $H(\Omega)$ that is not principal. Hint: For a sequence of points consider functions that vanish on all but finitely many.

11. Show that there is a maximal ideal that is not finitely generated. Hint: Zorn's lemma says that every proper ideal is contained in a maximal ideal.