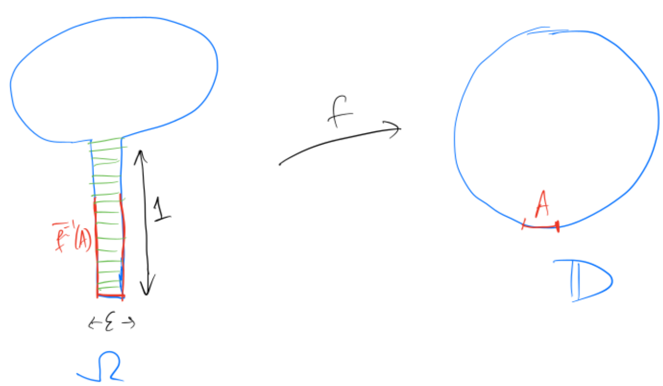
Homework 11: Riemann mappings and normal families

Riemann mappings. {1}

1. Let $\Omega \subset \mathbb{C}$ be the domain pictured, a disk with a long and skinny rectangle attached at the bottom. Let $f : \Omega \to \mathbb{D}$ be the (inverse of the) Riemann map, and let $\overline{f} : \overline{\Omega} \to \overline{\mathbb{D}}$ be the homeomorphism extending f (from the Carathéodory-Osgood theorem). Show that there is an arc $A \subset \partial\mathbb{D}$ whose endpoints are at distance $\leq 2\sqrt{\pi\epsilon}$ and so that $\overline{f}^{-1}(A)$ contains the lower half of the boundary of the rectangle portion of Ω. So if you traverse $\partial\mathbb{D}$ with constant speed, the image in $\partial\Omega$ under $\overline{f}^{-1}$ will speed up dramatically when it's in the lower half of the rectangle. Hint: Mimic the proof of Carathéodory-Osgood and show that the images of horizontal green arcs in the rectangle will on average have length $\leq \sqrt{\pi\epsilon}$.



Maximum principle.

Should have assigned these a few weeks ago. Better late than never.

2. Let Ω be a bounded domain and $f_n : \overline{\Omega} \to \mathbb{C}$ continuous and holomorphic on Ω. Show that if f_n converge uniformly on $\partial\Omega$ then they converge uniformly on $\overline{\Omega}$.

3. Let $z_1, z_2, \cdots, z_n$ be on the unit circle $\{|z| = 1\}$. Show that there is some $z \in \{|z| = 1\}$ such that the product of the distances to all z_k is ≥ 1.

Normal families.

4. Show that the family of functions $z \mapsto z^n$, $n = 1, 2, \cdots$ is normal on $\{|z| < 1\}$ but not on any larger domain.

5. Let $\mathcal{F}$ be a normal family of holomorphic functions $\Omega \to \mathbb{C}$ for a domain Ω. Show that the family of derivatives

$$\mathcal{F}' = \{f' \mid f \in \mathcal{F}\}$$

is also normal.

6. Suppose $\mathcal{F}$ is a normal family of functions defined on $\mathbb{D}$. Show that there are constants C_n so that if $f \in \mathcal{F}$ and $f(z) = \sum a_n z^n$ is the power series of f around 0, then $|a_n| \leq C_n$ for all n.

7. Let $\Omega \subset \mathbb{C}$ be a bounded domain (not necessarily simply connected) and $f : \Omega \to \Omega$ injective holomorphic. If $f(a) = a$ and $f'(a) = 1$ for some $a \in \Omega$, show that f is identity. Hint: Assuming $a = 0$ and $f \neq id$ write $f(z) = z + bz^n + \cdots$ around 0. Argue that $\{f^k \mid k = 1, 2, \cdots\}$ is a normal family and write the beginning of a power series around 0 for f^k to see that b must be 0, using Problem 6. There is also a proof using the Schwarz lemma if we grant the Uniformization theorem, but the intended argument is with normal families.

8. Let $\Omega \subset \mathbb{C}$ be a domain. For a holomorphic function $f : \Omega \to \mathbb{C}$ define the L^2-norm of f to be

$$\|f\|_{L^2(\Omega)} = \left(\int_\Omega |f|^2 d(\text{area}) \right)^{1/2}$$

   (i) For $\Omega = D(0, r)$ (disk centered at 0 and radius r) show that

$$|f(0)| \leq \frac{1}{r\sqrt{\pi}} \|f\|_{L^2(D(0,r))}$$

   and deduce that for any Ω and any compact set $K \subset \Omega$ there is a constant C_K so that

$$\max_{z \in K} |f(z)| \leq C_K \|f\|_{L_2}$$

   Thus the L^∞-norm is bounded by (a multiple of) the L^2-norm.

(ii) Show that

$$\{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic and } \|f\|_{L^2} \leq 1\}$$

is a normal family.

(iii) Suppose f is holomorphic on $\Omega = \mathbb{D} \setminus \{0\}$. If $\|f\|_{L^2(\Omega)} < \infty$ show that 0 is a removable singularity.

Hints and comments: This is a small variant of an old qual question and seems quite a bit harder than most qual questions. For (i) use the method as in Carathéodory-Osgood (or Problem 1 in this set). Start with $f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\theta})d\theta$. You'd like the area form $\rho\, d\rho d\theta$ in the integral, so first multiply by ρ and then integrate over $\rho \in (0, r)$. Then use Cauchy-Schwarz to replace $|f(z)|d(\text{area})$ by $|f(z)|^2 d(\text{area})$. I don't know what normality has to do with (iii), but you can use the criterion that 0 is removable if $\lim_{z\to 0} zf(z) = 0$, and the estimate from (i) applied to smaller and smaller disks with 0 in the boundary (the L^2-norm on these disks goes to 0 by Lebesgue dominated convergence).