Homework 5: Singularities, Residue Calculus

Singularities

1. If f is a holomorphic function defined in $\{|z| > R\}$ (we think of this set as a neighborhood of ∞) we say that ∞ is a removable or essential singularity or a pole provided that 0 is the respective singularity for the function $g(z) = f(\frac{1}{z})$. Show that

   (i) A nonconstant polynomial has a pole at infinity.

   (ii) If f is an entire function which is not a polynomial, then f has an essential singularity at ∞.

   Note that our standard essential singularities such as $e^{\frac{1}{z}}$ or $\sin \frac{1}{z}$ come from the construction in (ii), and we also get some new examples, e.g. $e^{\frac{1}{z}+\frac{1}{z^2}}$ etc.

Residue Calculus.

2. Let P be a polynomial of degree ≥ 2.

   (a) Show that for any circle C of big enough radius so that it encloses all roots we have
   $$\int_C \frac{dz}{P(z)} = 0$$

   (b) Assuming all roots $z_1, \cdots, z_n$ of P are distinct prove (using the Residue theorem) that
   $$\sum_{j=1}^{n} \frac{1}{P'(z_j)} = 0$$

   You may want to spend a few minutes thinking about how to prove (ii) without the Residue theorem.

3. Compute
   $$\frac{i}{4} \int_{|z|=2023} \tan(\pi z) dz$$

4. Compute
   $$\frac{1}{2\pi i} \int_{|z|=1} \sin\left(\frac{1}{z}\right) dz$$

5. Show that
$$\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{2\pi}{|1 - r^2|}$$
when $r \in \mathbb{R} \setminus \{-1, 1\}$.

6. Prove the Wallis formula
$$\frac{1}{2\pi} \int_0^{2\pi} (2\cos\theta)^{2m} d\theta = \binom{2m}{m}$$
for $m = 1, 2, \cdots$.

7. Prove that
$$\int_0^{\infty} \frac{dx}{1 + x^n} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$
for $n = 2, 3, \cdots$.

Hint: Use the contour consisting of $[0, R], [0, Re^{2\pi i/n}]$ and the short arc of $|z| = R$ connecting the endpoints.

8. Prove that
$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + 1)^2} dx = \frac{\pi}{2e}$$
Hint: As usual, $f(z) = \frac{z^3 e^{iz}}{(z^2+1)^2}$.

9. Prove that
$$\int_0^{\infty} \frac{x \sin x}{x^4 + 1} dx = \frac{\pi}{2} e^{-1/\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}}\right)$$

10. Prove that
$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x} dx = \frac{\pi}{\cosh\left(\frac{\pi}{2}\right)}$$

Hint: For the contour take the rectangle of height π and base $[-R, R]$.

11. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$
This can be done in many ways, including Fourier series, but here you should use Residue Calculus.

12. Prove that
$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12}$$
Hint: $\pi \csc \pi z = \frac{\pi}{\sin \pi z}$ has residue $(-1)^n$ at $z = n$.