Homework 4: Runge, zeros, Laurent

Runge

1. In class we constructed a sequence of polynomials that pointwise converges to a discontinuous function. Find a variant of this construction to show that there is a sequence of polynomials that pointwise converges to the zero function on $\mathbb{C}$, but not uniformly in any neighborhood of 0.

Zeros.

2. Show that the only holomorphic function f on the unit disk such that $f(\frac{1}{n}) = 0$ for $n = 2, 3, \cdots$ is the zero function.

3. Show that there are holomorphic functions f other than the zero function on the punctured disk $\{z \mid |z| < 1, z \neq 0\}$ such that $f(\frac{1}{n}) = 0$ for $n = 2, 3, \cdots$.

4. Show that there are no holomorphic functions f on the unit disk such that $f(\frac{1}{n}) = e^{-n}$ for $n = 2, 3, \cdots$.

5. Show that the only holomorphic function f on the unit disk such that $|f(\frac{1}{n})| \leq e^{-n}$ for $n = 2, 3, \cdots$ is the zero function. Hint: If not, write $f(z) = z^m g(z)$ with $g(0) \neq 0$.

6. Suppose f is holomorphic in $|z| < 2$ and for every $n = 2, 3, \cdots$

$$\int_{|z|=1} \frac{f(z)}{nz - 1} dz = 0$$

   Show that f is the zero function.

7. What can you say about f if the displayed equation is replaced with

$$\int_{|z|=1} \frac{f(z)}{(nz - 1)^2} dz = 0$$

8. Is there a holomorphic function f on the unit disk such that

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{n}$$

   for $n = 2, 3, \cdots$?

9. Suppose f is an entire function and for every $z_0 \in \mathbb{C}$ the power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

around z_0 has at least one coefficient a_n equal to zero. Show that f is a polynomial.

Laurent series.

10. Let $a, b \in \mathbb{C}$ with $0 < |a| < |b|$ and let

$$f(z) = \frac{1}{(z-a)(z-b)}$$

Find the Laurent series expansions of f in

(a) $|z| < |a|$,

(b) $|a| < |z| < |b|$,

(c) $|z| > |b|$.

11. Let α, β be two disjoint simple closed curves in $\mathbb{C}$ such that the disk A bounded by α is contained in the disk B bounded by β. Let Ω be the domain (annulus) $int(B \setminus A)$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that f can be written uniquely as

$$f = g + h$$

where g is holomorphic in $int(B)$, h is holomorphic in $\mathbb{C} \setminus \overline{A}$ and $g(z) \to 0$ as $z \to \infty$. This generalizes the case when α, β are concentric round circles, when g corresponds to the part of the Laurent series with nonnegative powers of z.