Homework 3: Cauchy, Morera, Integrals

Cauchy inequalities, Liouville

1. Suppose f is an entire function (holomorphic on all of $\mathbb{C}$) that satisfies $|f(z)| \leq A|z|^n + B$ for some n, A, B. Show that f is a polynomial.

2. Suppose that f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is a complex number λ such that $f(z) = \lambda g(z)$ for all $z \in \mathbb{C}$. Warning: If you consider f/g you should argue that it is well-defined at the zeros of g.

3. Suppose f_n is a sequence of holomorphic functions on a domain Ω that converges pointwise to a function f. Assuming all f_n are uniformly bounded on each compact subset of Ω show that convergence is uniform on compact sets. Hint: Cauchy for $f_n - f_m$ plus a named theorem from real analysis. Comment: False without assuming uniform boundedness, as we shall see later.

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence ≥ 1. Suppose that $|f'(z)| \leq 1$ for all z with $|z| < 1$. Prove that $|a_n| \leq \frac{1}{n}$ for all n. By example, show that these inequalities are sharp.

Morera's theorem.

5. Prove the following version of Morera's theorem. Suppose $f : \Omega \to \mathbb{C}$ is continuous and Ω is the open rectange $\{z \in \mathbb{C} \mid Re(z) \in (-1, 1), Im(z) \in (-1, 1)\}$. Suppose that $\int_\gamma f(z)dz = 0$ for every rectangle γ in Ω with sides parallel to the real and imaginary axes. Show that f is holomorphic. Note: The assumption that Ω is a rectangle is just for convenience; the statement is true for any open set.

Integrals. In the first three problems use the same curve we used in class, consisting of segments $[-R, -\epsilon]$, $[\epsilon, R]$ and the two semicircles centered at 0 of radius ϵ and R.

6. (Dirichlet integral) $\int_0^\infty \frac{\sin x}{x} dx$. Hint: e^{iz}/z.

7. $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$. Hint: $\frac{1-e^{2iz}}{z^2}$. Actually, this integral is equivalent to the one we did in class after a simple substitution.

8. Prove that $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{4}$. Hint: $\frac{3e^{iz} - e^{3iz}}{z^3}$.

9. Compute the Fresnel integrals

$$\int_{-\infty}^{\infty} \cos(t^2)dt \qquad \text{and} \qquad \int_{-\infty}^{\infty} \sin(t^2)dt$$

You can use the Gaussian integral calculation

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

from real analysis. Hint: Consider $f(z) = e^{-z^2}$ and integrate on the sector that consists of segments $[0, R]$, $[0, Re^{i\pi/4}]$ and the circular arc connecting R and $Re^{i\pi/4}$.