Homework 1: Cauchy-Riemann equations, power series

Cauchy-Riemann equations.
Let $\Omega \subset \mathbb{C} = \mathbb{R}^2$ be a domain (connected open set).

1. Suppose $\phi, \psi : \Omega \to \mathbb{R}$ are differentiable and satisfy Cauchy-Riemann equations
$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$
Show that pairs of functions
   (a) $\phi_1 = \phi^2 - \psi^2$, $\psi_1 = 2\phi\psi$,
   (b) $\phi_2 = e^{\phi} \cos \psi$, $\psi_2 = e^{\phi} \sin \psi$,
   (c) $\phi_3 = -\psi$, $\psi_3 = \phi$ (defined on $i\Omega$)

   also satisfy the Cauchy-Riemann equations. In each case represent the holomorphic function $f_j = \phi_j + i\psi_j$ directly in terms of $f = \phi + i\psi$.

2. Suppose $f : \Omega \to \mathbb{C}$ is holomorphic. If any of the following functions are constant, prove that f is constant:
   (a) $Re(f)$,
   (b) $Im(f)$,
   (c) $|f|$.

   Later in the course we will learn better ways to do this (Open Mapping Theorem), but here you should use Cauchy-Riemann equations.

3. A function $u : \Omega \to \mathbb{R}$ is called *harmonic*[1] if $u = Re(f)$ for some holomorphic function $f : \Omega \to \mathbb{C}$.
   (a) Show that $Im(f)$ is also harmonic when f is holomorphic.
   (b) If $u : \Omega \to \mathbb{R}$ is harmonic, up to adding a constant there is a unique $v : \Omega \to \mathbb{R}$ such that $u + iv$ is holomorphic.
   (c) Suppose that $u : \Omega \to \mathbb{R}$ is harmonic and has continuous second partial derivatives (this is always true as we shall see later). Show that u satisfies the Laplace equation
$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

[1] strictly speaking, being harmonic is a local property and the definition I am giving is correct only when Ω is simply connected

Power series.

4. Prove that $e^{z+w} = e^z e^w$ for any $z, w \in \mathbb{C}$.

5. Prove that $sin^2 z + cos^2 z = 1$ for any $z \in \mathbb{C}$.

6. Show that $\sum_n nz^n$ has radius of convergence 1 but does not converge for *any* z on the unit circle.

7. Show that $\sum_n z^n/n^2$ has radius of convergence 1 and converges for *every* z on the unit circle (in fact, absolutely and uniformly on $\{|z| \leq 1\}$).

8. An *arithmetic sequence* is a sequence of integers of the form $a, a + d, a + 2d, \cdots$ with $d > 0$ called the *step*. Show that $\mathbb{N} = \{1, 2, 3, \cdots\}$ cannot be partitioned into finitely many subsets each of which forms an arithmetic sequence with pairwise distinct steps. Hint: Otherwise write $1/(1-z)$ as the sum of finitely many functions of the form $z^a/(1-z^d)$.

Comment: In Problems 4 and 5 the intent is to manipulate the defining power series, using the fact that they are absolutely convergent. In the next few lectures an easier argument will emerge, based on the fact that if two holomorphic functions defined on $\mathbb{C}$ agree on $\mathbb{R}$ then they agree everywhere. Then #5 is immediate, and for #4 you can do it in two steps. First fix $w \in \mathbb{R}$ and argue that $z \mapsto e^{z+w}$ and $z \mapsto e^z e^w$ are equal. Then repeat this with $w \in \mathbb{C}$.

Abel's theorem and convergence on the circle $|z| = R$.

The goal of the next exercise is to show that it is legitimate to plug in $z = 1$ in the power series

- $\log(1 + z) = z - z^2/2 + z^3/3 - \cdots$
- $\arctan z = z - z^3/3 + z^5/5 - \cdots$

even though the radius of convergence is 1, thus proving that

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log 2$$

and

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \pi/4$$

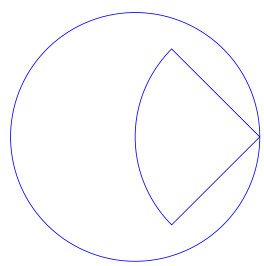
The key property is that the series happens to converge to *something* for $z = 1$.

9. (a) Let $a_0, a_1, \cdots$ and $b_0, b_1, \cdots$ be two sequences of complex numbers and set $s_n = a_0 + a_1 + \cdots + a_n$. Prove *Abel's summation by parts* for $n \geq 0$, $p \geq 1$:

$$\sum_{k=n+1}^{n+p} a_k b_k = \sum_{k=n+1}^{n+p} s_k(b_k - b_{k+1}) - s_n b_{n+1} + s_{n+p} b_{n+p+1}$$

Notice the similarity with $\int u dv = uv - \int v du$.

(b) Now let $\sum a_n z^n$ be a power series with radius of convergence 1 and assume that $\sum a_n$ converges. Abel's theorem states that convergence of $\sum a_n z^n$ is uniform (though not necessarily absolute) on any closed sector centered at 1 and contained in $\{|z| < 1\}$ except for the center. In particular, the limiting function is continuous on the sector. This justifies the validity of plugging in $z = 1$. Pictured is a sector with bounding lines of slope ± 1.



Prove Abel's theorem. It might make it easier to assume the sector is just $[0, 1]$, which is sufficient for the examples.

Hint: First show that there is a constant $M > 0$ so that $|1 - z| \leq M(1 - |z|)$ on the sector (e.g. $M = 1$ for the degenerate sector $[0, 1]$). To simplify the situation, add a constant so that $\sum a_n = 0$. Take $b_n = z^n$ in Abel's summation. For any $\epsilon > 0$ choose n so that $|s_k| < \epsilon$ for $k \geq n$. Then estimate the main sum by $\epsilon|1 - z||z|^n \frac{1}{1-|z|}$.

10. This might be called the anti-Abel's theorem. Suppose $a_0 \geq a_1 \geq a_2 \geq \cdots \to 0$ and assume $\sum a_n z^n$ has radius of convergence 1. Example: $a_n = 1/n$. Thus the series may not converge for $z = 1$. However, show that it converges for any other z with $|z| = 1$. In fact, show that the convergence is uniform on any set of the form $\{z \mid |z| \leq 1, |1 - z| \geq \delta\}$ for any $\delta > 0$. Hint: Abel's summation again, this time set it up so

that the a_k from the summation is z^k and b_k is a_k from the series (sorry about the notational conflict).

11. Let $z_1, z_2, \cdots, z_p$ be distinct points on the unit circle. Show that the following power series with radius of convergence 1 diverges for $z = z_1, z_2, \cdots, z_p$ but converges at all other points on the unit circle:

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{z_1^n} + \frac{1}{z_2^n} + \cdots + \frac{1}{z_p^n} \right) z^n$$