Homework 8: Möbius transformations, Riemann surfaces

Notation: $\mathbb{D} = \{z| \mid |z| < 1\}$ and $\mathbb{H} = \{z \mid Im(z) > 0\}$.

Möbius transformations.

1. Determine which of the following transformations are elliptic, parabolic, loxodromic. Recall that I don't distinguish between "hyperbolic" and "loxodromic" (if you are looking at Ahlfors).

   (i) $z \mapsto z/(2z-1)$.

   (ii) $z \mapsto 2z/(3z-1)$.

   (iii) $z \mapsto (3z-4)/(z-1)$.

   (iv) $z \mapsto z/(2-z)$.

2. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $f(0) = a$ show that f has no zeros in $\{|z| < |a|\}$. Hint: Find an automorphism of $\mathbb{D}$ that interchanges 0 and a and apply the Schwarz lemma.

Riemann surfaces

3. In class we have seen that the unit disk $\mathbb{D}$ and the upper half-plane $\mathbb{H}$ are biholomorphically (or conformally) equivalent, and they are both equivalent to the strip $\{z \mid 0 < Im(z) < \pi\}$. Show that the following domains in $\mathbb{C}$ are also conformally equivalent to these three. Note: The Riemann mapping theorem says that every simply connected domain, not all of $\mathbb{C}$, is conformally equivalent to $\mathbb{D}$. The point here is that the conformal equivalence is realized by explicit maps.

   (i) The sector $\{z \mid 0 < arg(z) < \alpha\}$ for any $\alpha \in (0, 2\pi]\}$.

   (ii) Upper half-plane with a slit $\mathbb{H} \setminus [0, i]$. Hint: z^2.

   (iii) Upper half-plane with a circular slit $\mathbb{H} \setminus \{z \mid |z| = 1, Re(z) \geq 0\}$.

   (iv) Half-strip $\{z \mid Re(z) < 0, 0 < Im(z) < \pi\}$. Hint: e^z.

4. Show that $\mathbb{C} \setminus \{0\}$ and $\mathbb{D} \setminus \{0\}$ are not conformally equivalent. Hint: Liouville.

5. Recall from class that
$$X(a,b,c,\infty) = \{(z,w) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}} \mid w^2 = (z-a)(z-b)(z-c)\}$$
is a Riemann surface for $a, b, c \in \mathbb{C}$ distinct. It is a double branched cover over $\hat{\mathbb{C}}$ branched over a, b, c, ∞. Show that if the cross-ratios (a, b, c, ∞) and (a', b', c', ∞) are equal after possibly permuting the points, then $X(a, b, c, \infty)$ and $X(a, b, c, \infty)$ are conformally equivalent. Start by observing that the permutation can be taken to fix ∞. Hint 1: This is really an exercise in covering spaces. Hint 2: It can also be an exercise in pull-backs. There is a diagram in the shape of a cube with the front and back faces pull-back diagrams that define $X(a, b, c, \infty)$ and $X(a', b', c', \infty)$, all faces are commutative, and maps relating front and back are (biholomorphic) homeomorphisms. Note: The surface $X(a, b, c, \infty)$ is a torus, and we will see that if the cross-ratios are different then the tori are not biholomorphic. In fact, the "moduli space" of tori (or "elliptic curves" in algebraic geometry) is the space of quadruples of distinct points in the Riemann sphere with the cross-ratio equivalence. Tori, along with annuli, have simplest interesting moduli spaces.

6. This problem is more involved than most. In the class we really considered $X = X(0, 1, 2, \infty)$. The projection to the second coordinate $X \to \hat{\mathbb{C}}$, $(z, w) \mapsto w$, is holomorphic, hence a branched cover. Compute branch points, their preimages, and the local degree at each. To start, compute all this for the map $\hat{C} \to \hat{C}$ given by the polynomial $z(z-1)(z-2)$. Warning: Unlike the projection to z, after removing branch points and their primages the resulting covering map is not regular.

A "well-known" identity.

The goal here is to prove that the "principal" sum
$$\sum_{n\in\mathbb{Z}} \frac{1}{z-n} = \pi \cot(\pi z)$$
The sum doesn't converge in the usual sense, just like the harmonic series doesn't, but if we group the terms for n and $-n$ it does:

7. Show that $f(z) = \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}$ converges absolutely for every $z \notin \mathbb{Z}$, and f is a meromorphic function on $\mathbb{C}$ with simple poles at $\mathbb{Z}$ and residue 1 at each.

8. Show that $f(z+1) = f(z)$ for every z. Hint: $f(z)$ is the limit of $\sum_{n=-N}^{N} \frac{1}{z-n}$.

9. Let X be the Riemann surface $\mathbb{C}/z \sim z+1$, i.e. a cyclinder. Thus f defines a holomorphic function $F : X \to \hat{\mathbb{C}}$. Let $\hat{X}$ be the compactification of X biholomorphic to $\hat{\mathbb{C}}$, with two ideal points $\pm\infty$ (corresponding to imaginary part going to $\pm\infty$) as in Problem 6. Show that F extends to a holomorphic function $\hat{F} : \hat{X} \to \hat{\mathbb{C}}$ via $\hat{F}(\pm\infty) = \mp\pi i$. Hint: It suffices to extend continuously. When $z = Mi$ this amounts to computing $-2Mi \sum_{n\geq 1} \frac{1}{M^2+n^2}$. The sum can be estimated with $\int_0^\infty \frac{1}{M^2+x^2} dx$, which can be computed via calculus or calculus of residues.

10. Show that $g(z) = \pi \cot(\pi z)$ also has the following properties: it is meromorphic with simple poles at $\mathbb{Z}$ and residue 1 at each, $g(z+1) = g(z)$, and the induced map G on the cylinder X extends to a function $\hat{G}$ on the compactification with the same values at $\pm\infty$.

11. Show that this implies $f = g$. Hint: Show that $\hat{F}\hat{G}^{-1}$ is a Möbius transformation that fixes three points.

There is a more direct way to do this, but I think it involves more calculations. Namely, using the method to solve Basel's problem, we can calculate $f(z)$ for all z. The advantage of the method outlined above is that the only real work is calculating $f(\pm\infty)$, the rest is citing theorems about the Riemann sphere and Möbius transformations.