Homework 5: Singularities, Residue Calculus

Singularities

1. If f is a holomorphic function defined in $\{|z| > R\}$ (we think of this set as a neighborhood of ∞) we say that ∞ is a removable or essential singularity or a pole provided that 0 is the respective singularity for the function $g(z) = f(\frac{1}{z})$. Show that

   (i) A nonconstant polynomial has a pole at infinity.

   (ii) If f is an entire function which is not a polynomial, then f has an essential singularity at ∞.

Residue Calculus.

2. Let P be a polynomial of degree ≥ 2.

   (a) Show that for any circle C of big enough radius so that it encloses all roots we have
   $$\int_C \frac{dz}{P(z)} = 0$$

   (b) Assuming all roots $z_1, \cdots, z_n$ of P are distinct prove (using the Residue theorem) that
   $$\sum_{j=1}^{n} \frac{1}{P'(z_j)} = 0$$

3. Compute
$$\frac{1}{2\pi i} \int_{|z|=1} \sin\left(\frac{1}{z}\right) dz$$

4. Show that
$$\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{2\pi}{|1 - r^2|}$$
when $r \in \mathbb{R} \setminus \{-1, 1\}$.

5. Prove the Wallis formula
$$\frac{1}{2\pi} \int_0^{2\pi} (2\cos\theta)^{2m} d\theta = \binom{2m}{m}$$
for $m = 1, 2, \cdots$.

6. Prove that
$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\frac{\pi}{n}}{\sin\left(\frac{\pi}{n}\right)}$$
for $n = 2, 3, \cdots$.

   Hint: Use the contour consisting of $[0, R], [0, Re^{2\pi i/n}]$ and the short arc of $|z| = R$ connecting the endpoints.

7. Prove Jordan's lemma: Let F be a continuous function defined in $\{z \in \mathbb{C} \mid |z| \geq R_0, Im(z) \geq 0\}$ satisfying $F(z) \to 0$ when $z \to \infty$. If $m > 0$ and Γ_R denotes the semicircle $\{z \mid |z| = R, Im(z) \geq 0\}$, then
$$\int_{\Gamma_R} e^{imz} F(z)\, dz \to 0 \qquad \text{as } R \to \infty$$

   Hint: When the dust settles you will want to show that $R \int_0^\pi e^{-mRsin(t)} dt$ is uniformly bounded as $R \to \infty$. It is convenient to break the integral into $\int_0^{\pi/2}$ and $\int_{\pi/2}^{\pi}$. E.g. on the first one argue that for some $\epsilon > 0$ we have $sin(t) \geq \epsilon t$ on $[0, \pi/2]$ and then you can calculate the integral with ϵt in place of $sin(t)$.

8. Prove that
$$\int_{-\infty}^\infty \frac{x^3 \sin x}{(x^2+1)^2} dx = \frac{\pi}{2e}$$
   Hint: As usual, $f(z) = \frac{z^3 e^{iz}}{(z^2+1)^2}$.

9. Prove that
$$\int_0^\infty \frac{x \sin x}{x^4+1} dx = \frac{\pi}{2} e^{-1/\sqrt{2}} \sin\left(\frac{1}{\sqrt{2}}\right)$$

10. Prove that
$$\int_{-\infty}^\infty \frac{\cos x}{\cosh x} dx = \frac{\pi}{\cosh\left(\frac{\pi}{2}\right)}$$
    Hint: For the contour take the rectangle of height π and base $[-R, R]$.

11. Prove that
$$\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$$
    This can be done in many ways, including Fourier series, but here you should use Residue Calculus.

12. Prove that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12}$$

Hint: $\pi \csc \pi z = \frac{\pi}{\sin \pi z}$ has residue $(-1)^n$ at $z = n$.

13. We showed in class that if $\phi : C \to \mathbb{C}$ is a continuous function defined on the circle $|z| = 1$ then the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\phi(\xi)}{\xi - z} \, d\xi$$

defines a holomorphic function in $\mathbb{D} = \{|z| < 1\}$.

(i) Compute f when $\phi(\xi) = \overline{\xi} = 1/\xi$.

(ii) Compute f when $\phi(\xi) = \xi^n$ for $n \in \mathbb{Z}$.

(iii) Compute f when ϕ is given by its Fourier series

$$\phi(\xi) = \sum_{n \in \mathbb{Z}} a_n \xi^n$$

under the assumption that convergence of the series is uniform.

Hint: The answer to (iii) is to remove the negative powers. So from this point of view holomorphic functions form a "half-dimensional" subspace of the space of functions on the circle.