Homework 2: Integration along paths

This homework set is more along the lines of "routine calculations", but the goal is to get comfortable doing these kinds of calculations as there will be many of them throughout the semester.

Recall that a *primitive* for a function f is a function F such that $F' = f$. In this situation, the Fundamental Theorem of Calculus states that $\int_\gamma f(z)dz = F(w_2) - F(w_1)$ if γ is a piecewise smooth path from w_1 to w_2.

Several problems here use the Estimation Theorem:

$$\left| \int_\gamma f(z)dz \right| \leq L(\gamma) \max_{x \in \gamma} |f(z)|$$

where $L(\gamma)$ is the length of γ.

1. Evaluate the following integrals $\int_\gamma f(z)dz$. Where possible, use the existence of primitives and the Fundamental Theorem of Calculus. Problem (iii) can be reduced to the Fundamental Calculation we did in class.

   (i) $f(z) = z^2$, $\gamma(t) = e^{it}$, $t \in [-\pi/2, \pi/2]$.

   (ii) $f(z) = e^z$, γ is the join of the segments $[0, 1]$, $[1, 1+i]$ and $[1+i, i]$.

   (iii) $f(z) = 1/z$, $\gamma(t) = e^{-it}$, $t \in [0, 8\pi]$.

2. Evaluate the following integrals $\int_\gamma f(z)dz$.

   (i) $f(z) = Re(z)$, $\gamma(t) = t + it^2$, $t \in [0, 1]$.

   (ii) $f(z) = |z|^4$, γ is the interval $[-1 + i, 1 + i]$.

3. Evaluate the following integrals $\int_\gamma f(z)dz$ where $\gamma = \gamma(0, 1)$ is the circle centered at 0 with radius 1 oriented counterclockwise. Where possible, use the Fundamental Calculation from class or the Fundamental Theorem of Calculus. For (i) recall that the integral depends only on the values of f along γ, and maybe you can find a better function g such that $g = f$ on γ.

   (i) $f(z) = |z|^4$.

   (ii) $f(z) = z^{-2}(z^4 - 1)$.

   (iii) $f(z) = \sin z$.

4. Use the Estimation Theorem to prove the following upper bounds. We denote by $\gamma(c,r)$ the circle of radius r centered at c.

   (i) $|\int_{\gamma(1,2)} \frac{1}{z} dz| \leq 4\pi$.

   (ii) $|\int_{\gamma(0,r)} \frac{z-1}{z+1} dz| \leq \frac{2\pi r(r+1)}{|r-1|}$.

   (iii) $|\int_{\Gamma_R} \frac{e^{iz}}{z^4} dz| \leq \pi R^{-3}$, where Γ_R is the semicircle

   $$\Gamma_R = \gamma(0,R) \cap \{z \mid Im(z) \geq 0\}$$

   (iv) $|\int_{[0,1+i]} (z^2+1)^{-1} dz| \leq \sqrt{2}$.

5. (a) Let $p(z)$ be a polynomial. Show that for any closed path γ in $\mathbb{C}$

   $$\int_\gamma p(z) dz = 0$$

   (b) Show that for some $\epsilon > 0$ the following holds. If $p(z)$ is any polynomial then there is some z_0 with $|z_0| = 1$ such that $|p(z_0) - 1/z_0| \geq \epsilon$. Thus the function $1/z$ cannot be uniformly approximated by polynomials on the unit circle. Find the best possible ϵ and for that ϵ find an approximating polynomial.

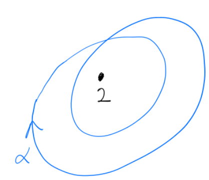
Cauchy integral formula

In the following problems, $f(z) = \frac{e^{z^2}}{z-2}$. This is holomorphic on $\mathbb{C} \setminus \{2\}$. You should use Cauchy's theorem or integral formula to compute the following integrals. $\gamma(a,r)$ denotes the circle centered at a with radius r, oriented counterclockwise.

6. $\int_{\gamma(2+i,2)} f(z)\ dz$

7. $\int_{\gamma(3+i,1)} f(z)\ dz$

8. $\int_\alpha f(z)\ dz$, where α is the curve shown.



Cauchy inequalities

9. Suppose f is an entire function (holomorphic on all of $\mathbb{C}$) that satisfies $|f(z)| \leq A|z|^n + B$ for some integer $n \geq 0$ and $A, B > 0$. Show that f is a polynomial.

Power series

Just so we don't forget power series, here is another problem involving them.

10. Prove that for $|z| < 1$ the following identity holds. Justify any manipulations with power series.

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z}$$