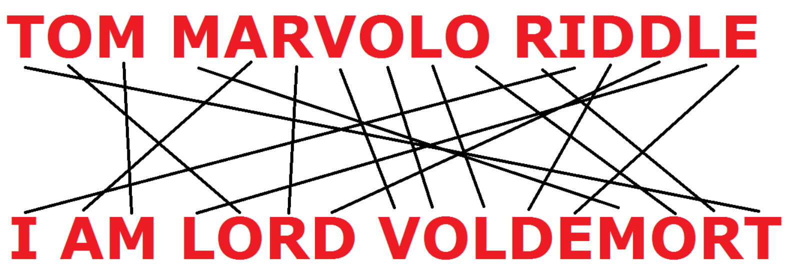
Permutations

This handout assumes you are familiar with Artin 1.5.

1. Let σ be a permutation of the set $\{1, 2, \cdots, n\}$. Consider the set of pairs (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. Thus, these are the pairs whose order is "flipped" by σ. A good way to visualize σ is to draw a $2 \times n$ array of dots and join the $(1, i)$ dot with the $(2, \sigma(i))$ dot by a segment. Then (i, j) is 'flipped" if the two segments leaving $(1, i)$ and $(1, j)$ intersect. We say that the permutation σ is *even*, and write $\mathrm{sign}(\sigma) = 1$, if the number of flipped pairs is even. Likewise, we say σ is *odd* and write $\mathrm{sign}(\sigma) = -1$ if the number of flipped pairs is odd. So we "only" need to count the number of intersection points and look at its parity.



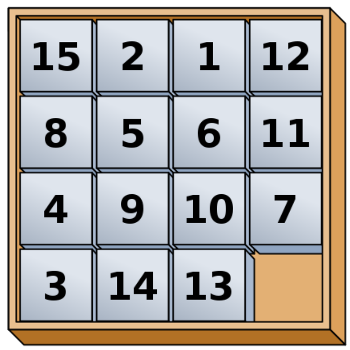
2. Find the sign of the permutation $(12)(345)$
3. Show that every transposition is odd.
4. Any permutation and its inverse have the same sign.
5. The cycle $(12 \cdots k)$ is even for k odd, and odd for k even. In fact, all odd cycles are even permutations and even cycles are odd permutations (!) but that's a harder excercise.
6. For which n is the permutation

$$1 \mapsto n, 2 \mapsto n-1, \cdots, i \mapsto n+1-i, \cdots, n \mapsto 1$$

even?

7. Show that $\mathrm{sign}(\sigma\sigma') = \mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma')$.

8. Recall the "15-puzzle", a 4×4 square with 15 square tiles numbered $1, 2, \cdots, 15$ and with one empty square. You are allowed to slide a tile to the empty square next to it. The goal of the puzzle is to arrange the tiles so they appear in order when read left to right from the top row to the bottom row, with the lower right square empty. Prove that if this position is achievable, then the position where 1 and 2 are transposed and the other numbers are where they should be is impossible.



Hint for #7: Draw 3 rows of dots and picture σ' on the top two rows, and σ on the bottom two rows. Then show that (i, j) is flipped for $\sigma\sigma'$ iff the concatenated segments leaving $(1, i)$ and $(1, j)$ intersect in one point, and otherwise they intersect in 0 or 2 points.