Determinants

This handout is about determinants. I don't like the way Artin and other books define them. I think the sign of a permutation is a much more basic/elementary concept than the determinant and defining it in terms of determinants, as Artin does, is putting the cart before the horse.

1. Let $A = \big(a_{ij}\big)$ be an $n \times n$ matrix. Define the determinant

   $$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

   where the summation goes over all permutations of $\{1, 2, \cdots, n\}$.

2. Just as a sanity check, show that for $n = 1, 2, 3$ we get the usual formulas for the determinant.

3. Show that the determinant of a permutation matrix (look up the definition in Artin) is equal to the sign of the permutation.

4. The determinant of an upper (lower) triangular matrix is the product of the diagonal entries.

5. Show that we would get the same formula if we reversed the roles of rows and colums and wrote

   $$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$$

   Hint: A permutation and its inverse have the same sign.

6. If a row (column) is 0, the determinant is 0.

7. If two rows are swapped, the determinant changes the sign.

8. If two rows (columns) are equal, determinant is 0.

9. Multiplying a row/column by x multiplies the determinant by x.

10. If A, B, C are 3 matrices that have all entries equal except in one, say i^{th}, row (or column), and in that row we have $c_{ij} = a_{ij} + b_{ij}$ for all j, then $\det(C) = \det(A) + \det(B)$. Thus the determinant is a linear function of a row, when the other rows are fixed.

11. Adding a multiple of one row/column to another does not change the determinant. Hint: This follows from #9,10.

12. $\det(AB) = \det(A)\det(B)$. Hint: First prove this when one of the two matrices is an elementary matrix.

13. A is invertible iff $\det(A) \neq 0$.

14. Prove directly from our definition of the determinant that Artin's formula holds, say using the first row.

15. The summand $a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$ can be pictured by placing n rooks on the $n \times n$ chess board. No two rooks attack each other and any such placement of rooks corresponds to a summand. Is there a quick way to determine the sign of the permutation from the position of the rooks? Maybe even find the cycle decomposition?