2022 FALL MATH 5310 HOMEWORK 9 SOLUTIONS
DUE: OCT 31ST

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Question 1 (Artin 4.7.1)**.** Determine the Jordan form of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution. Observe the characteristic polynomial of A is $p(t) = (t-1)^3$, so we take $B = A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ to simplify to the case when the eigenvalue is 0. Note $B^2 = 0$, so choosing a vector like $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ such that $Bv = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \neq 0$, we get a linearly independent set (v, Bv). This gives us a 2×2 Jordan block of B. For the last basis, we append a vector $w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \ker B$ that is not contained in $\operatorname{span}\langle v, Bv \rangle$, which gives a 1×1 (zero) Jordan block.

All in all, with the basis $(v, Bv, w) = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$, the Jordan form of B is $J_B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so the Jordan form of A is $J_A = J_B + I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. //

Question 2 (Artin 4.7.2)**.** Prove that $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ is an *idempotent* matrix, i.e., that $A^2 = A$, and find its Jordan form.

Solution. That $A^2 = A$ can be checked from computation. Note that this implies that every generalized eigenvector of A for 0 is indeed an eigenvector for 0, as $A^n v = Av = 0$ for every $n \geq 1$ and an eigenvector v of A for 0. From the characteristic polynomial of A being $p(t) = t^2(t-1)$, the Jordan form for $\lambda = 0$ consists of two a 1×1 Jordan block, which is just 2×2 Jordan matrix. For $\lambda = 1$, it can only have the 1×1 Jordan block, so the resulting Jordan form of A is:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with the basis (v_1, v_1', v_2) where (v_1, v_1') is a basis for the eigenspace of $\lambda = 0$, and v_2 is an eigenvector for $\lambda = 1$. //

Question 3 (Artin 4.7.3)**.** Let V be a complex vector space of dimension 5, and let T be a linear operator on V whose characteristic polynomial is $(t-\lambda)^5$. Suppose that the rank of the operator $T - \lambda I$ is 2. What are the possible Jordan forms for T?

Solution. Since the rank of $T - \lambda I$ is 2, the kernel of $T - \lambda I$ has dimension 3. Hence, Jordan forms for T consist of three blocks, so there are two possible Jordan forms: $(3,1,1)$ or $(2,2,1)$:

$$\begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{bmatrix} \quad \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & & \lambda & & \\ & & 1 & \lambda & \\ & & & & \lambda \end{bmatrix}$$

up to permuting the Jordan blocks. //

Question 4 (Artin 4.7.7)**.** Is every complex square matrix A such that $A^2 = A$ diagonalizable?

Solution. Yes. Say $A^2 = A$ and J be the Jordan form of A, such that $J = P^{-1}AP$ for some P. Then $J^2 = P^{-1}A^2P = P^{-1}AP = J$. Since J is a block diagonal matrix with Jordan blocks, it follows that $J_i^2 = J_i$ for each Jordan block J_i of J. Decompose $J_i = D_i + N_i$, as a diagonal part $D_i = \lambda I$ and nilpotent part N_i. Then

$$J_i^2 = D_i^2 + 2D_iN_i + N_i^2 = \lambda^2 I + 2\lambda N_i + N_i^2 = J_i = \lambda I + N_i.$$

For the diagonal entries to be matched, we have $\lambda^2 = \lambda$, so it follows that $\lambda = 0$ or $\lambda = 1$. Plugging in $\lambda = 0, 1$ into above equation, we have $N_i^2 = \pm N_i$. Since N_i is nilpotent, this implies that $N_i = 0$. Therefore, each Jordan block J_i has to have only diagonal component with $\lambda = 0$ or $\lambda = 1$, which implies that J is a diagonal matrix, so A is diagonalizable. //

Question 5 (Artin 4.7.8)**.** Is every complex square matrix A similar to its transpose?

Solution. Yes. Say J is the Jordan form of A and $A = P^{-1}JP$ for some P. Then $A^t = P^tJ^t(P^t)^{-1}$, so it suffices ot prove J^t is similar to J. Write $J = D + N$ as a diagonal-nilpotent decomposition. Since $J^t = D + N^t$ as D is symmetric, and we have $Q^{-1}J^tQ = D + Q^{-1}N^tQ$, it further reduces to proving N and N^t are similar. Reducing once more, as each nilpotent part can be decomposed as nilpotent Jordan blocks $N = N_1 \oplus \ldots \oplus N_k$ and $N^t = N_1^t \oplus \ldots \oplus N_k^t$, it is sufficient to proving each nilpotent block N_i is similar to N_i^t. Say N_i is of $d \times d$. Then N_i represents a nilpotent linear operator T_i with the standard basis $(e_1, \ldots, e_d)$. Now observe that by changing the basis to $(e_d, \ldots, e_1)$, N_i^t represents the same linear operator T_i. To be precise, letting

$$P = \begin{bmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

then $P^{-1}N_iP = N_i^t$. (Note $P^2 = I$, so $P^{-1} = P$.) This proves N_i and N_i^t are similar for each Jordan block N_i of N, concluding the proof. //

Question 6 (Bonus: Artin 4.7.4).

(1) Determine all possible Jordan forms for a matrix whose characteristic polynomial is $(t+2)^2(t-5)^3$.

(2) What are the possible Jordan forms for a matrix whose characteristic polynomial is $(t+2)^2(t-5)^3$, when space of eigenvectors with eigenvalue -2 is one-dimensional, and the space of eigenvectors with eigenvalue 5 is two-dimensional?

Solution. (a) First, as the characteristic polynomial is of degree 5, the matrix should be of size 5×5. From the observation that Jordan forms are triangular, the Jordan form should contain two -2's and three 5's in diagonal. Therefore, the possible Jordan forms are:

$$\begin{bmatrix} -2 & & & & \\ & -2 & & & \\ & & 5 & & \\ & & & 5 & \\ & & & & 5 \end{bmatrix}, \begin{bmatrix} -2 & & & & \\ & -2 & & & \\ & & 5 & & \\ & & 1 & 5 & \\ & & & & 5 \end{bmatrix}, \begin{bmatrix} -2 & & & & \\ & -2 & & & \\ & & 5 & & \\ & & 1 & 5 & \\ & & & 1 & 5 \end{bmatrix},$$

$$\begin{bmatrix} -2 & & & & \\ 1 & -2 & & & \\ & & 5 & & \\ & & & 5 & \\ & & & & 5 \end{bmatrix}, \begin{bmatrix} -2 & & & & \\ 1 & -2 & & & \\ & & 5 & & \\ & & 1 & 5 & \\ & & & & 5 \end{bmatrix}, \begin{bmatrix} -2 & & & & \\ 1 & -2 & & & \\ & & 5 & & \\ & & 1 & 5 & \\ & & & 1 & 5 \end{bmatrix},$$

up to permuting Jordan blocks.

(b) Since the eigenspace for eigenvalue -2 is one-dimensional, the Jordan block for -2 should be $\begin{bmatrix} -2 & \\ 1 & -2 \end{bmatrix}$. Also, knowing the eigenspace for 5 is two-dimensional, we can rule out $\begin{bmatrix} 5 & & \\ 1 & 5 & \\ & 1 & 5 \end{bmatrix}, \begin{bmatrix} 5 & & \\ & 5 & \\ & & 5 \end{bmatrix}$ from the candidates for the Jordan block for 5. In summary, the only possible Jordan form is:

$$\begin{bmatrix} -2 & & & & \\ 1 & -2 & & & \\ & & 5 & & \\ & & 1 & 5 & \\ & & & & 5 \end{bmatrix}. \quad //$$